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To: Board of Education Members

From: Walter Wakeman
PK-5 Mathematics & Science Curriculum Leader

Date: March 22, 2013

Subject: Response to BOE Requests Regarding Mathematics

Attached you will find documents as requested by Board of Education members regarding the mathematics curriculum. Those include:

- Research and articles of reference
- Sample of grade 4 resources with supporting materials
- List of assessments administered to preschool-grade 2
- Student achievement results
- PK-12 continuum of mathematics domains and standards
- 2013-2014 proposed mathematics budget

# *Research and Articles for Reference*

The attached research and articles are a sampling of what was used to guide the development of our curriculum, instructional model and District Textbook Review Committee work.

**Mathematical Proficiency**
-*Adding It Up* – National Research Council, 2001
Researchers identified five mathematical proficiencies to capture what they think it means to be mathematically proficient: conceptual understanding, procedurally fluency, strategic competence, adaptive reasoning, and productive disposition. "One of the most serious and persistent problems facing school mathematics in the United State is the tendency to concentrate on one strand of proficiency to the exclusion of the rest." p. 11

**What Is Important in Early Childhood Mathematics**
-National Council of Teachers of Mathematics position, 2007
It is important to engage young children in meaningful mathematics in deep and sustained ways.

**What's All this Talk About Rigor?**
-National Council of Teachers of Mathematics President Linda M. Gojak, 2013
This article explains what rigor looks like in a mathematics classroom.

**Big Ideas and Understandings as a Foundation for Elementary and Middle School Mathematics**
-National Council of Supervisors of Mathematics Journal Spring Summer, 2005
Big Ideas in mathematics inform the development of the instructional model and coherence in the units in grades 3 – 5.

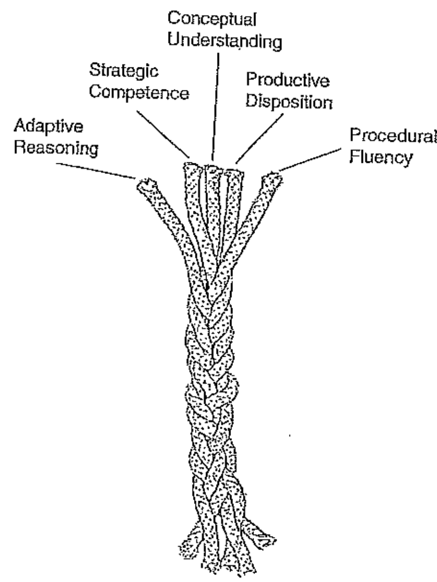
**K-8 Publishers' Criteria for the Common Core State Standards for Mathematics**
-National Governors Association et al., 2012
This article provides a set of criteria when reviewing resources to align with the Common Core State Standards.

**Common Core State Standards (CCSS) for Mathematics,** 2010
CCSS were adopted by Connecticut in July of 2010.

## Mathematical Proficiency

Our analyses of the mathematics to be learned, our reading of the research in cognitive psychology and mathematics education, our experience as learners and teachers of mathematics, and our judgment as to the mathematical knowledge, understanding, and skill people need today have led us to adopt a composite, comprehensive view of successful mathematics learning. Recognizing that no term captures completely all aspects of expertise, competence, knowledge, and facility in mathematics, we have chosen *mathematical proficiency* to capture what we think it means for anyone to learn mathematics successfully. Mathematical proficiency, as we see it, has five strands:

- *conceptual understanding*—comprehension of mathematical concepts, operations, and relations
- *procedural fluency*—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- *strategic competence*—ability to formulate, represent, and solve mathematical problems
- *adaptive reasoning*—capacity for logical thought, reflection, explanation, and justification
- *productive disposition*—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy.




Intertwined Strands of Proficiency

The most important observation we make about these five strands is that they are interwoven and interdependent. This observation has implications for how students acquire mathematical proficiency, how teachers develop that proficiency in their students, and how teachers are educated to achieve that goal.

## The Mathematical Knowledge Children Bring to School

Children begin learning mathematics well before they enter elementary school. Starting from infancy and continuing throughout the preschool period, they develop a base of skills, concepts, and misconceptions. At all ages, students encounter quantitative situations outside of school from which they learn a variety of things about number. Their experiences include, for example, noticing that a sister received more candies, counting the stairs

Adding It Up, National Research Council, 2001.

## Developing Proficiency in Teaching Mathematics

Proficiency in teaching mathematics is related to effectiveness: consistently helping students learn worthwhile mathematical content. It also entails versatility: being able to work effectively with a wide variety of students in different environments and across a range of mathematical content. Despite the common myth that teaching is little more than common sense or that some people are just born teachers, effective teaching practice can be learned. Just as mathematical proficiency itself involves interwoven strands, teaching for mathematical proficiency requires similarly interrelated components: *conceptual understanding* of the core knowledge of mathematics, students, and instructional practices needed for teaching; *procedural fluency* in carrying out basic instructional routines; *strategic competence* in planning effective instruction and solving problems that arise while teaching; *adaptive reasoning* in justifying and explaining one's practices and in reflecting on those practices; and a *productive disposition* toward mathematics, teaching, learning, and the improvement of practice.

Effective programs of teacher preparation and professional development help teachers understand the mathematics they teach, how their students learn that mathematics, and how to facilitate that learning. In these programs, teachers are not given prescriptions for practice or readymade solutions to teaching problems. Instead, they adapt what they are learning to deal with problems that arise in their own teaching.

## Recommendations

The overriding premise of our work is that throughout the grades from pre-K through 8 all students can and should be mathematically proficient.

As a goal of instruction, mathematical proficiency provides a better way to think about mathematics learning than narrower views that leave out key features of what it means to know and be able to do mathematics. It takes time for proficiency to develop fully, but in every grade in school, students can demonstrate mathematical proficiency in some form. **The overriding premise of our work is that throughout the grades from pre-K through 8 *all students can and should be mathematically proficient.***

School mathematics in the United States does not now enable most students to develop the strands of mathematical proficiency in a sound fashion. Proficiency for all demands that fundamental changes be made concurrently in curriculum, instructional materials, assessments, classroom practice, teacher preparation, and professional development. These changes will require continuing, coordinated action on the part of policy makers, teacher educators, teachers, and parents. Although some readers may feel that substantial advances are already being made in reforming mathematics teaching and learning, we find real progress toward mathematical proficiency to be woefully inadequate.

These observations lead us to five principal recommendations regarding mathematical proficiency that reflect our vision for school mathematics. The full report augments these five with specific recommendations that detail policies and practices needed if all children are to become mathematically proficient.

- **The integrated and balanced development of all five strands of mathematical proficiency (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition) should guide the teaching and learning of school mathematics. Instruction should not be based on extreme positions that students learn, on one hand, solely by internalizing what a teacher or book says or, on the other hand, solely by inventing mathematics on their own.**

One of the most serious and persistent problems facing school mathematics in the United States is the tendency to concentrate on one strand of proficiency to the exclusion of the rest. For too long, students have been the victims of crosscurrents in mathematics instruction, as advocates of one learning goal or another have attempted to control the mathematics to be taught and tested. We believe that this narrow and unstable treatment of mathematics is, in part, responsible for the inadequate performance that U.S. students display on national and international assessments. Our first recommendation is that these crosscurrents be resolved into an integrated, balanced treatment of all strands of mathematical proficiency at every point in teaching and learning.

Although we endorse no single approach, we contend that instruction needs to configure the relations among teachers, students, and mathematics in ways that promote the development of mathematical proficiency. Under this view, significant instructional time is devoted to developing concepts and methods, and carefully directed practice, with feedback, is used to support learning. Discussions build on students' thinking. They attend to relationships between problems and solutions and to the nature of justification and mathematical argument. All strands of proficiency can grow in a coordinated, interactive fashion.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

**What Is Important in Early Childhood Mathematics?**
A Position of the National Council of Teachers of Mathematics

**Question**
Why is mathematics important for early childhood learners?

**NCTM Position**
The National Council of Teachers of Mathematics affirms that a high-quality, challenging, and accessible mathematics education provides early childhood learners with a vital foundation for future understanding of mathematics. Young children in every setting should experience effective, research-based curricula and teaching practices. Such practices in turn require policies, organizational support, and resources that enable teachers to do this challenging and important work.

Increasing numbers of young children are in settings where they can encounter mathematics in experiences that build on one another, expanding early understanding sequentially, in developmentally appropriate ways. Research on children's learning in the first six years of life validates the importance of early experiences in mathematics for lasting positive outcomes. A growing body of research also supports curricular resources for early mathematics. Teacher preparation programs, education agencies, policymakers, and other partners must commit resources and mobilize to support teachers and collaborate in developing effective early childhood mathematics programs.

In a high-quality mathematics program for early childhood learners, teachers and caregivers can enhance children's natural interest in mathematics and their instinct to use it to organize and make sense of their world. Mathematical experiences for young children should take advantage of familiar contexts, building on relationships within families, linguistic and cultural backgrounds, and the informal knowledge of early learners. Mathematics curricula and teaching practices should rest on a solid understanding of both mathematics and the development of young children.

Teaching practices should strengthen young children's problem-solving and reasoning abilities in experiences that are both informal and involve more formal, prepared materials. Teachers should connect ideas within mathematics as well as with other subjects, and they should encourage children to communicate, explaining their thinking as they interact with important mathematics in deep and sustained ways. Finally, early childhood educators should actively introduce mathematical concepts, methods, and language through a range of appropriate experiences and teaching strategies. These should be monitored by observation and other informal evaluations to ensure that instructional decisions are based on each child's mathematical needs.

Teacher education programs should give attention to the mathematics component of early childhood programs, and continuing professional development opportunities should support high-quality mathematics education. The development of institutional policies that promote teachers' mathematical learning, teamwork, and planning can provide the necessary resources to overcome the classroom, community, institutional, and system-wide barriers to young children's mathematical proficiency. Such initiatives will ensure the future of young children, who are our next generation of mathematics learners.

September 2007

# What's All This Talk about Rigor?

By NCTM President Linda M. Gojak
NCTM *Summing Up*, February 5, 2013

Recently, I had a conversation with a group of math coaches who are working with elementary teachers on implementation of the Common Core Standards for Mathematics. The discussion turned to a description of rigor in the classroom. The coaches commented that many of their teachers were confused by exactly what was meant by teaching and learning with rigor. The coaches weren't sure how to respond.

**Rigor in the Common Core State Standards**

The word "rigor" is widely used in policy discussions, but it's rarely understood or defined, and often it merely passes as code for "better." It is interesting that the term "rigor" does not appear in the Common Core State Standards for Mathematics, although it is certainly implied. "Rigor" appears multiple times in the U.S. Department of Education's paper, "A Blueprint for Reform: The Reauthorization of the Elementary and Secondary Education Act," as well as its recent document, "ESEA Flexibility"—both of which include a call for rigorous academic content standards.

**Rigor in Instruction**

The coaches and I began our work of exploring the notion of rigor with an online search of the word "rigor." The thesaurus led us to a list of synonyms, including "affliction," "inflexibility," "difficulty," "severity," "rigidity," "suffering," and "traditionalism"—none of which describe characteristics of rigorous mathematics instruction. No wonder the teachers were confused! However, two additional words included in the list—"thoroughness"and "tenacity"—provided avenues for some serious thought about what "rigor"implies. We generated the following chart, which led to an interesting discussion with the classroom teachers. There are certainly other characteristics that can be added to the list.

| **Learning experiences that involve rigor ...** | **Experiences that do not involve rigor ...** |
|---|---|
| challenge students | are more "difficult," with no purpose (for example, adding 7ths and 15ths without a real context) |
| require effort and tenacity by students | require minimal effort |
| focus on quality (rich tasks) | focus on quantity (more pages to do) |
| include entry points and extensions for all students | are offered only to gifted students |
| are not always tidy, and can have multiple paths to possible solutions | are scripted, with a neat path to a solution |
| provide connections among mathematical ideas | do not connect to other mathematical ideas |
| contain rich mathematics that is relevant to students | contain routine procedures with little relevance |
| develop strategic and flexible thinking | follow a rote procedure |
| encourage reasoning and sense making | require memorization of rules and procedures without understanding |
| expect students to be actively involved in their own learning | often involve teachers doing the work while students watch |

**Rigor Involves Everyone**

Rigor involves all partners in teaching and learning. Teachers must consider rigor in planning lessons, tasks, and assignments. Rigorous lessons build on and extend prior knowledge. They encourage productive struggling. Although the objective of a lesson should be clear in the teacher's mind, the lesson should not focus on one correct path to a solution or even one correct answer. A rigorous lesson embraces the messiness of a good mathematics task and the deep learning that it has the potential to achieve.

Students who are successful in a rigorous learning environment take responsibility for their learning. They learn to reflect on their thinking. They persist in solving a problem when the path to solution is not immediately obvious. They recognize when they are not on the correct path and need to switch directions during the solution process. Students must learn to ask productive questions rather than expecting to be shown how to proceed. (And, teachers must answer those questions with just enough information to move students forward while preserving the challenge of the task!

Rigorous teaching and learning require rigorous formative assessment throughout a unit so the teacher knows what the student has learned and can plan additional activities, or adjust them, to address student needs. Students also have a role in formative assessment—they must approach tasks with tenacity and ask clarifying questions when they are unsure how to proceed. All assessments must include opportunities for students to demonstrate the processes and practices in their approach to doing mathematics. Good formative assessment can be incorporated into daily instruction and prepare students for the summative assessments that take place at certain points throughout the unit of study.

**Moving toward Rigor**

How can we support classroom teachers and pre-service teachers (pre-K–16) in working toward greater rigor in mathematics instruction? Professional development experiences that model rigor through the use of rich tasks, rich discourse, and good questions allow teachers to experience rigorous instruction. When selecting tasks, teachers must be sure that mathematical ideas are explicit and the connections are clear. The days of a few word problems at the end of multiple skill exercises in the textbook are over! Concepts must be introduced and explored in contexts that are interesting and motivating for students. Tasks must provide entry points for all students, offer them well-defined opportunities to make connections to other mathematics, and include both opportunities and expectations for them to develop deeper understanding. The focus and coherence of the Common Core State Standards lead the way to rigorous instruction. It is time for us to begin the journey.

# Big Ideas and Understandings as the Foundation for Elementary and Middle School Mathematics

Randall I. Charles, Carmel, CA

Education has always been grounded on the principle that high quality teaching is directly linked to high achievement and that high quality teaching begins with the teacher's deep subject matter knowledge. Mathematics education in the United States has been grounded on this principle, and most educators and other citizens have always believed that our teachers have adequate content knowledge given the high mathematics achievement of our students. Unfortunately, research conducted in the past ten years has shown that the United States is not among the highest achieving countries in the world, and that our teacher's subject matter knowledge and teaching practices are fundamentally different than those of teachers in higher achieving countries.

Research is beginning to identify important characteristics of highly effective teachers (Ma 1999, Stigler 2004; Weiss, Heck, and Shimkus, 2004). For example, effective teachers ask appropriate and timely questions, they are able to facilitate high-level classroom conversations focused on important content, and they are able to assess students' thinking and understanding during instruction. Another, and the focus of this paper, is the grounding of a teacher's mathematics content knowledge and their teaching practices around a set of Big Mathematical Ideas *(Big Ideas)*.

The purpose of this paper is to initiate a conversation about the notion of Big Ideas in mathematics. Although Big Ideas have been talked about for some time, they have not become part of mainstream conversations about mathematics standards, curriculum, teaching, learning, and assessment. Given the growing evidence as to their importance, it is timely to start these conversations. A definition of a Big Idea is presented here along with a discussion of their importance. Then a set of Big Ideas and Understandings for elementary and middle school mathematics is proposed. The paper closes with some suggestions for ways Big Ideas can be used.

In working with colleagues on the development of this paper I am rather certain that it is not possible to get one set of Big Ideas and Understandings that all mathematicians and mathematics educators can agree on. Fortunately, I do not think it's necessary to reach a consensus in this regard. Use the Big Mathematical Ideas and Understandings presented here as a starting point for the conversations they are intended to initiate.

## What is a *Big Idea* in mathematics?

*Teachers need to understand the big ideas of mathematics and be able to represent mathematics as a coherent and connected enterprise.* (NCTM, 2000, p. 17)

Teachers are being encouraged more and more through statements such as the one above to teach to the big ideas of mathematics. Yet if you ask a group of teachers or any group of mathematics educators for examples of big ideas, you'll get quite a variety of answers. Some will suggest a topic, like equations, others will suggest a strand, like geometry, others will suggest an expectation, such as those found in *Principles and Standards for School Mathematics* (NCTM, 2000), and some will even suggest an objective, such as those found in many district and state curriculum standards. Although all of these are important, none seems sufficiently robust to qualify as a big idea in mathematics. Below is a proposed definition of a big idea, and it is the one that was used for the work shared in this paper.

**DEFINITION: A *Big Idea* is a statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understandings into a coherent whole.**

There are several important components of this definition. First, a Big Idea is a statement; here's an example.

**Any number, measure, numerical expression, algebraic expression, or equation can be represented in an infinite number of ways that have the same value.**

For ease of discussion each Big Idea below is given a word or phrase before the statement of the Big Idea (e.g., Equivalence). It is important to remember that this word or phrase is a *name* for the Big Idea; it is not the idea itself. Rather the Big Ideas are the statements that follow the name. Articulating a Big Idea as a statement forces one to come to grips with the essential mathematical meaning of that idea.

The second important component of the definition of a Big Idea given above is that it is an *idea central to the learning of mathematics*. For example, there are many mathematical concepts (e.g., number, equality, numeration) and there are many mathematical processes (e.g., solving linear equations using inverse operation and properties of equality) where understanding is grounded on knowing that mathematical objects like numbers, expressions, and equations can be represented in different ways without changing the value or solution, that is, equivalence. Also, knowing the kinds of changes in representations that maintain the same value or the same solution is a powerful problem-solving tool.

Ideas central to the learning of mathematics can be identified in different ways. One way is through the careful analysis of mathematics concepts and skills; a content analysis that looks for connections and commonalities that run across grades and topics. This approach was used to develop the Big Ideas presented here drawing on the work of others who have articulated ideas central to learning mathematics (see e.g., NCTM 1989, 1992, 2000; O'Daffer and others, 2005; Van de Walle 2001). Some additional thoughts are given later about identifying Big Ideas.

The third important component of the definition of a Big Idea is that it *links numerous mathematics understandings into a coherent whole*. Big Ideas make connections.[1] As an example, the early grades curriculum introduces several "strategies" for figuring out basic number combinations such as 5 + 6 and 6 x 7. The strategy of use a double involves thinking that 5 + 6 is the same as 5 + 5 and 1 more. The strategy of use a five fact involves thinking that 6 x 7 is the same as 5 x 7 and 7 more. Both of these strategies, and others, are connected through the idea of equivalence; both involve breaking the calculation apart into an equivalent representation that uses known facts to figure out the unknown fact. Good teaching should make these connections explicit.

A set of Big Ideas for elementary and middle school are given later in this paper. For each Big Idea examples of *mathematical understandings* are given. A *mathematical understanding* is an important idea students need to learn because it contributes to understanding the Big Idea. Some mathematical understandings for Big Ideas can be identified through a careful content analysis, but many must be identified by "listening to students, recognizing common areas of confusion, and analyzing issues that underlie that confusion" (Schifter, Russell, and Bastable 1999, p. 25). Research and classroom experience are important vehicles for the continuing search for mathematical understandings.

## Why are Big Ideas Important?

Big Ideas should be the foundation for one's mathematics content knowledge, for one's teaching practices, and for the mathematics curriculum. Grounding one's mathematics content knowledge on a relatively few Big Ideas establishes a robust understanding of mathematics. Hiebert and his colleagues say, "We understand something if we see how it is related or connected to other things we know" (1997, p. 4), and "The degree of understanding is determined by the number and strength of the connections" (Hiebert & Carpenter, 1992, p. 67). Because Big Ideas have connections to many other ideas, understanding Big Ideas develops a deep understanding of mathematics. When one understands Big Ideas, mathematics is no longer seen as a set of disconnected concepts, skills, and facts. Rather, mathematics becomes a coherent set of ideas. Also, understanding Big Ideas has other benefits.

[1] This attribute of a big idea is consistent with definitions others have provided- see Clements & DiBiase 2001; Ritchhart 1999; Southwest Consortium for the Improvement of Mathematics and Science Teaching 2002; Trafton & Reys, 2004.

Understanding:

- is motivating.
- promotes more understanding.
- promotes memory.
- influences beliefs.
- promotes the development of autonomous learners.
- enhances transfer.
- reduces the amount that must be remembered.

(Lambdin 2003).

Teachers who understand the Big Ideas of mathematics translate that to their teaching practices by consistently connecting new ideas to Big Ideas and by reinforcing Big Ideas throughout teaching (Ma 1999). Also, effective teachers know how Big Ideas connect topics across grades; they know the concepts and skills developed at each grade and how those connect to previous and subsequent grades.

And finally, Big Ideas are important in building and using curricula. *The Curriculum Principle from the Principles and Standards for School Mathematics* (NCTM, 2000) gives three attributes of a powerful curriculum.

1) A mathematics curriculum should be coherent.
2) A mathematics curriculum should focus on important mathematics.
3) A mathematics curriculum should be well articulated across the grades.

The National Research Council reinforced these ideas about curriculum: "...it is important that states and districts avoid long lists [of standards] that are not feasible and that would contribute to an unfocused and shallow mathematics curriculum" (2001, p. 35). By the definition given above, Big Ideas provide curriculum coherence and articulate the important mathematical ideas that should be the focus of curriculum.

## What are Big Ideas for elementary and middle school mathematics?

Twenty-one (21) Big Mathematical Ideas for elementary and middle school mathematics are given at the end of this paper. Knowing the process I used to develop this list and some issues I confronted in developing it might be helpful if you decide to modify it or build your own.

As part of a Kindergarten through Grade 8 curriculum development project, several colleagues and I articulated "math understandings" for every lesson we wrote in the program. Using the long list of math understandings we created, I organized these across content strands rather than grade levels. When I did that, it became apparent that there were clusters of math understandings, ideas that seem to be connected to something bigger. I then started the process of trying to articulate what it was that connected these ideas; I developed my definition of a Big Idea and used that as a guide. I next confronted a fundamental issue in doing this kind of work — how big (or small) is a Big Idea? Although I am not presumptuous enough to suggest an answer to this question, I can share some thinking that guided me. My sense is that Big Ideas need to be big enough that it is relatively easy to articulate several related ideas, what I called mathematical understandings. I also believe that Big Ideas need to be useful to teachers, curriculum developers, test developers, and to those responsible for developing state and district standards. If a Big Idea is too big, my sense is that its usefulness for these audiences diminishes. This thinking led to an initial list of 31 Big Ideas grouped into the traditional content strands. Reviews by colleagues suggested that articulating Big Ideas by content strands was not necessary; Big Ideas are BIG because many run across strands. This led to a reduction in the number of Big Ideas on my list. Further analyses of my list with regard to their usefulness for the audiences mentioned above led to the list offered in this paper.

Finally, it is important to note that there are relatively few Big Ideas in this list – this is what makes the notion of Big Ideas so powerful. One's content knowledge, teaching practices, and curriculum can all be grounded on a small number of ideas. This not only brings everything together for the teacher but most importantly it enables students to develop a deep understanding of mathematics.

## What are some ways Big Ideas can be used?

Here are a few ways that Big Mathematical Ideas and Understandings can be used.

### Curriculum Standards and Assessment

- Revise/create district and state curriculum standards to incorporate Big Mathematical Ideas and Understandings. Many state standards emphasize mathematical skills. Curriculum coherence and effective mathematics instruction starts with standards that embrace not just skills but also big mathematical ideas and understandings.

- Develop individual teacher, district, state, or national assessments around Big Mathematical Ideas and Understandings. Alignment of standards and assessment is important for many reasons and both need to address big ideas, understandings, and skills.

### Professional Development

- Build *professional development courses* focused on mathematics content and anchored on Big Ideas and Understandings. Engage teachers with tasks that enable them to grapple with Big Ideas and Understandings.
- Do a *lesson study* where Big Ideas are used to connect content and teaching practices (See Takahashi & Yoshida, 2004).
- Develop *chapter/unit and individual lesson plans* by starting with Big Ideas. Generate mathematical understandings specific to the content and grade level(s) of interest.

Appendix A shows an example of one way Big Ideas might be infused into an existing curriculum. In this example, the teachers did an analysis of all of the lessons in a fourth grade chapter on multiplication. Based on that analysis, they created a chapter overview that started with their state content and reasoning standards but then connected them to Big Ideas. Individual lessons were then connected to the Standards and Big Ideas and to the specific mathematics understandings to be developed in that lesson.

### Conclusion

The purpose of this paper is to start a conversation about Big Ideas. Use the Big Ideas and Math Understandings presented here as a starting point; edit, add, and delete as you feel best. But, as you develop your own set keep these points in mind. First, do not lose the essence of a Big Idea as defined here, and second, do not allow your list of Big Ideas and Understandings to balloon to a point where content and curriculum coherence are lost. Big Ideas need to remain BIG and they need to be the anchors for most everything we do.

## Big Mathematical Ideas and Understandings for Elementary and Middle School Mathematics

A *Big Idea* is a statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understandings into a coherent whole.

### BIG IDEA #1

**NUMBERS — The set of real numbers is infinite, and each real number can be associated with a unique point on the number line.**

**Examples of Mathematical Understandings:**

*Counting Numbers*

- Counting tells how many items there are altogether. When counting, the last number tells the total number of items; it is a cumulative count.
- Counting a set in a different order does not change the total.
- There is a number word and a matching symbol that tell exactly how many items are in a group.
- Each counting number can be associated with a unique point on the number line, but there are many points on the number line that cannot be named by the counting numbers.
- The distance between any two consecutive counting numbers on a given number line is the same.
- One is the least counting number and there is no greatest counting number on the number line.
- Numbers can also be used to tell the position of objects in a sequence (e.g., 3rd), and numbers can be used to name something (e.g., social security numbers).

*Whole Numbers*
- Zero is a number used to describe how many are in a group with no objects in it.
- Zero can be associated with a unique point on the number line.
- Each whole number can be associated with a unique point on the number line, but there are many points on the number line that cannot be named by the whole numbers.
- Zero is the least whole number and there is no greatest whole number on the number line.

*Integers*
- Integers are the whole numbers and their opposites on the number line, where zero is its own opposite.
- Each integer can be associated with a unique point on the number line, but there are many points on the number line that cannot be named by integers.
- An integer and its opposite are the same distance from zero on the number line.
- There is no greatest or least integer on the number line.

*Fractions/Rational Numbers*
- A fraction describes the division of a whole (region, set, segment) into equal parts.
- The bottom number in a fraction tells how many equal parts the whole or unit is divided into. The top number tells how many equal parts are indicated.
- A fraction is relative to the size of the whole or unit.
- A fraction describes division.( $a/b = a \div b$, a & b are integers & $b \neq 0$), and it can be interpreted on the number line in two ways. For example, $2/3 = 2 \div 3$. On the number line, $2 \div 3$ can be interpreted as 2 segments where each is 1/3 of a unit (2 x 1/3) or 1/3 of 2 whole units (1/3 x 2); each is associated with the same point on the number line. (Rational number)
- Each fraction can be associated with a unique point on the number line, but not all of the points between integers can be named by fractions.
- There is no least or greatest fraction on the number line.
- There are an infinite number of fractions between any two fractions on the number line.
- A decimal is another name for a fraction and thus can be associated with the corresponding point on the number line.
- Whole numbers and integers can be written as fractions (e.g., 4 = 4/1, -2 = -8/4).
- A percent is another way to write a decimal that compares part to a whole where the whole is 100 and thus can be associated with the corresponding point on the number line.
- Percent is relative to the size of the whole.

## BIG IDEA #2

**THE BASE TEN NUMERATION SYSTEM — The base ten numeration system is a scheme for recording numbers using digits 0-9, groups of ten, and place value.**

**Examples of Mathematical Understandings:**

*Whole Numbers*
- Numbers can be represented using objects, words, and symbols.
- For any number, the place of a digit tells how many ones, tens, hundreds, and so forth are represented by that digit.
- Each place value to the left of another is ten times greater than the one to the right (e.g., 100 = 10 x 10).
- You can add the value of the digits together to get the value of the number.
- Sets of ten, one hundred and so forth must be perceived as single entities when interpreting numbers using place value (e.g., 1 hundred is one group, it is 10 tens or 100 ones).

*Decimals*

- Decimal place value is an extension of whole number place value.
- The base-ten numeration system extends infinitely to very large and very small numbers (e.g., millions & millionths).

## BIG IDEA #3

**EQUIVALENCE: Any number, measure, numerical expression, algebraic expression, or equation can be represented in an infinite number of ways that have the same value.**

**Examples of Mathematical Understandings:**

*Numbers and Numeration*

- Numbers can be decomposed into parts in an infinite number of ways
- Numbers can be named in equivalent ways using place value (e.g., 2 hundreds 4 tens is equivalent to 24 tens).
- Numerical expressions can be named in an infinite number of different but equivalent ways (e.g., $4/6 \div 2/8 = 2/3 \div 1/4 = 2/3 \times 4/1$; also $26 \times 4 = (20 + 6) \times 4$).
- Decimal numbers can be named in an infinite number of different but equivalent forms (e.g., $0.3 = 0.30 = 0.10 + 0.20$).

*Number Theory and Fractions*

- Every composite number can be expressed as the product of prime numbers in exactly one way, disregarding the order of the factors (Fundamental Theorem of Arithmetic).
- Every fraction/ratio can be represented by an infinite set of different but equivalent fractions/ratios.

*Algebraic Expressions and Equations*

- Algebraic expressions can be named in an infinite number of different but equivalent ways (e.g., $2(x - 12) = 2x - 24 = 2x - (28 - 4)$).
- A given equation can be represented in an infinite number of different ways that have the same solution (e.g., $3x - 5 = 16$ and $3x = 21$ are equivalent equations; they have the same solution, 7).

*Measurement*

- Measurements can be represented in equivalent ways using different units (e.g., 2 ft 3 in = 27 in.).
- A given time of day can be represented in more than one way.
- For most money amounts, there are different, but finite combinations of currency that show the same amount; the number of coins in two sets does not necessarily indicate which of two sets has the greater value.

## BIG IDEA #4

**COMPARISON: Numbers, expressions, and measures can be compared by their relative values.**

**Examples of Mathematical Understandings:**

*Numbers & Expressions*

- One-to-one correspondence can be used to compare sets.
- A number to the right of another on the number line is the greater number.
- Numbers can be compared using greater than, less than, or equal.
- Three or more numbers can be ordered by repeatedly doing pair-wise comparisons.
- Whole numbers and decimals can be compared by analyzing corresponding place values.
- Numerical and algebraic expressions can be compared using greater than, less than, or equal.

*Fractions, Ratios, & Percent*

- A comparison of a part to the whole can be represented using a fraction.
- A ratio is a multiplicative comparison of quantities; there are different types of comparisons that can be represented as ratios.
- Ratios give the relative sizes of the quantities being compared, not necessarily the actual sizes.
- Rates are special types of ratios where unlike quantities are being compared.
- A percent is a special type of ratio where a part is compared to a whole and the whole is 100.
- The probability of an event is a special type of ratio.

*Geometry and Measurement*

- Lengths can be compared using ideas such as longer, shorter, and equal.
- Mass/weights can be compared using ideas such as heavier, lighter, and equal.
- Measures of area, volume, capacity and temperature can each be compared using ideas such as greater than, less than, and equal.
- Time duration for events can be compared using ideas such as longer, shorter, and equal.
- Angles can be compared using ideas such as greater than, less than, and equal.

## BIG IDEA #5

**OPERATION MEANINGS & RELATIONSHIPS: The same number sentence (e.g. 12-4 = 8) can be associated with different concrete or real-world situations, AND different number sentences can be associated with the same concrete or real-world situation.**

**Examples of Mathematical Understandings:**

*Whole Numbers*

- Some real-world problems involving joining, separating, part-part-whole, or comparison can be solved using addition; others can be solved using subtraction.
- Adding x is the inverse of subtracting x.
- Any subtraction calculation can be solved by adding up from the subtrahend.
- Adding quantities greater than zero gives a sum that's greater than any addend.
- Subtracting a whole number (except 0) from another whole number gives a difference that's less than the minuend.
- Some real-world problems involving joining equal groups, separating equal groups, comparison, or combinations can be solved using multiplication; others can be solved using division.
- Multiplying by x is the inverse of dividing by x.
- Any division calculation can be solved using multiplication.
- Multiplying two whole numbers greater than one gives a product greater than either factor.

*Rational Numbers (Fractions & Decimals)*

- The real-world actions for addition and subtraction of whole numbers are the same for operations with fractions and decimals.
- Different real-world interpretations can be associated with the product of a whole number and fraction (decimal), a fraction (decimal) and whole number, and a fraction and fraction (decimal and decimal).
- Different real-world interpretations can be associated with division calculations involving fractions (decimals).
- The effects of operations for addition and subtraction with fractions and decimals are the same as those with whole numbers.
- The product of two positive fractions each less than one is less than either factor.

*Integers*

- The real-world actions for operations with integers are the same for operations with whole numbers.

## BIG IDEA #6

**PROPERTIES: For a given set of numbers there are relationships that are always true, and these are the rules that govern arithmetic and algebra.**

**Examples of Mathematical Understandings:**

*Properties of Operations*

- Properties of whole numbers apply to certain operations but not others (e.g., The commutative property applies to addition and multiplication but not subtraction and division.).
- Two numbers can be added in any order; two numbers can be multiplied in any order.
- The sum of a number and zero is the number; the product of any non-zero number and one is the number.
- Three or more numbers can be grouped and added (or multiplied) in any order.

*Properties of Equality*

- If the same real number is added or subtracted to both sides of an equation, equality is maintained.
- If both sides of an equation are multiplied or divided by the same real number (not dividing by 0), equality is maintained.
- Two quantities equal to the same third quantity are equal to each other.

## BIG IDEA #7

**BASIC FACTS & ALGORITHMS: Basic facts and algorithms for operations with rational numbers use notions of equivalence to transform calculations into simpler ones.**

**Examples of Mathematical Understandings:**

*Mental Calculations*

- Number relationships and sequences can be used for mental calculations (one more, one less; ten more, ten less; 30 is two more than 28; counting back by thousands from 50,000 is 49,000, 48,000, 47,000 etc.)
- Numbers can be broken apart and grouped in different ways to make calculations simpler.

*Whole Number Basic Facts & Algorithms*

- Some basic addition and multiplication facts can be found by breaking apart the unknown fact into known facts. Then the answers to the known facts are combined to give the final value.
- Subtraction facts can be found by thinking of the related addition fact.
- Division facts can be found by thinking about the related multiplication fact.
- When 0 is divided by any non-zero number, the quotient is zero, and 0 cannot be a divisor.
- Addition can be used to check subtraction, and multiplication can be used to check division.
- Powers of ten are important benchmarks in our numeration system, and thinking about numbers in relation to powers of ten can make addition and subtraction easier.
- When you divide whole numbers sometimes there is a remainder; the remainder must be less than the divisor.
- The real-world situation determines how a remainder needs to be interpreted when solving a problem.

*Rational Number Algorithms*

- Fractions with unlike denominators are renamed as equivalent fractions with like denominators to add and subtract.
- The product of two fractions can be found by multiplying numerators and multiplying denominators.

- A fraction division calculation can be changed to an equivalent multiplication calculation (i.e., $a/b \div c/d = a/b \times d/c$, where b, c, and $d \neq 0$).
- Division with a decimal divisor is changed to an equivalent calculation with a whole number divisor by multiplying the divisor and dividend by an appropriate power of ten.
- Money amounts represented as decimals can be added and subtracted using the same algorithms as with whole numbers.

*Measurement*
- Algorithms for operations with measures are modifications of algorithms for rational numbers.
- Length measurements in feet and inches can be added or subtracted where 1 foot is regrouped as 12 inches.
- Times in minutes and seconds can be added and subtracted where 1 minute is regrouped as 60 seconds.

## BIG IDEA #8

**ESTIMATION: Numerical calculations can be approximated by replacing numbers with other numbers that are close and easy to compute with mentally. Measurements can be approximated using known referents as the unit in the measurement process.**

**Examples of Mathematical Understandings:**

*Numerical*
- The numbers used to make an estimate determine whether the estimate is over or under the exact answer.
- Division algorithms use numerical estimation and the relationship between division and multiplication to find quotients.
- Benchmark fractions like 1/2 (0.5) and 1/4 (0.25) can be used to estimate calculations involving fractions and decimals.
- Estimation can be used to check the reasonableness of exact answers found by paper/pencil or calculator methods.

*Measurement*
- Length, area, volume, and mass/weight measurements can be estimated using appropriate known referents.
- A large number of objects in a given area can be estimated by finding how many are in a sub-section and multiplying by the number of sub-sections.

## BIG IDEA #9

**PATTERNS: Relationships can be described and generalizations made for mathematical situations that have numbers or objects that repeat in predictable ways.**

**Examples of Mathematical Understandings:**

*Numbers*
- Skip counting on the number line generates number patterns.
- The structure of the base ten numeration system produces many numerical patterns.
- There are patterns in the products for multiplication facts with factors of 0, 1, 2, 5, and 9.
- There are patterns when multiplying or dividing whole numbers and decimals by powers of ten.
- The difference between successive terms in some sequences is constant.
- The ratio of successive terms in some sequences is a constant.
- Known elements in a pattern can be used to predict other elements.

*Geometry*
- Some sequences of geometric objects change in predictable ways.

## BIG IDEA #10

**VARIABLE: Mathematical situations and structures can be translated and represented abstractly using variables, expressions, and equations.**

**Examples of Mathematical Understandings:**

- Letters are used in mathematics to represent generalized properties, unknowns in equations, and relationships between quantities.
- Some mathematical phrases can be represented as algebraic expressions (e.g. Five less than a number can be written as $n - 5$.)
- Some problem situations can be represented as algebraic expressions (e.g. Susan is twice as tall as Tom; If $T$ = Tom's height, then $2T$ = Susan's height.)
- Algebraic expressions can be used to generalize some transformations of objects in the plane.

## BIG IDEA #11

**PROPORTIONALITY: If two quantities vary proportionally, that relationship can be represented as a linear function.**

**Examples of Mathematical Understandings:**

- A ratio is a multiplicative comparison of quantities.
- Ratios give the relative sizes of the quantities being compared, not necessarily the actual sizes.
- Ratios can be expressed as units by finding an equivalent ratio where the second term is one.
- A proportion is a relationship between relationships.
- If two quantities vary proportionally, the ratio of corresponding terms is constant.
- If two quantities vary proportionally, the constant ratio can be expressed in lowest terms (a composite unit) or as a unit amount; the constant ratio is the slope of the related linear function.
- There are several techniques for solving proportions (e.g., finding the unit amount, cross products).
- When you graph the terms of equal ratios as ordered pairs (first term, second term) and connect the points, the graph is a straight line.
- If two quantities vary proportionally, the quantities are either directly related (as one increases the other increases) or inversely related (as one increases the other decreases).
- Scale drawings involve similar figures, and corresponding parts of similar figures are proportional.
- In any circle, the ratio of the circumference to the diameter is always the same and is represented by the number pi.
- Rates can be related using proportions as can percents and probabilities.

## BIG IDEA #12

**RELATIONS & FUNCTIONS: Mathematical rules (relations) can be used to assign members of one set to members of another set. A special rule (function) assigns each member of one set to a unique member of the other set.**

**Examples of Mathematical Understandings:**

- Mathematical relationships can be represented and analyzed using words, tables, graphs, and equations.
- In mathematical relationships, the value for one quantity depends on the value of the other quantity.
- The nature of the quantities in a relationship determines what values of the input and output quantities are reasonable.
- The graph of a relationship can be analyzed with regard to the change in one quantity relative to the change in the other quantity.

- The graph of a relation can be analyzed to determine if the relation is a function.
- In a linear function of the form y = ax, a is the constant of variation and it represents the rate of change of y with respect to x.
- The solutions to a linear function form a straight line when graphed.
- A horizontal line has a slope of 0, and a vertical line does not have a slope.
- The parameters in an equation representing a function affect the graph of the function in predictable ways.

## BIG IDEA #13

**EQUATIONS & INEQUALITIES: Rules of arithmetic and algebra can be used together with notions of equivalence to transform equations and inequalities so solutions can be found.**

**Examples of Mathematical Understandings:**

- A solution to an equation is a value of the unknown or unknowns that makes the equation true.
- Properties of equality and reversible operations can be used to generate equivalent equations and find solutions.
- Techniques for solving equations start by transforming the equation into an equivalent one.
- A solution or solutions to a linear or quadratic equation can be found in the table of ordered pairs or from the graph of the related function.
- Techniques for solving equations can be applied to solving inequalities, but the direction of the inequality sign needs to be considered when negative numbers are involved.

## BIG IDEA #14

**SHAPES & SOLIDS: Two- and three-dimensional objects with or without curved surfaces can be described, classified, and analyzed by their attributes.**

**Examples of Mathematical Understandings:**

- Point, line, line segment, and plane are the core attributes of space objects, and real-world situations can be used to think about these attributes.
- Polygons can be described uniquely by their sides and angles.
- Polygons can be constructed from or decomposed into other polygons.
- Triangles and quadrilaterals can be described, categorized, and named based on the relative lengths of their sides and the sizes of their angles.
- All polyhedra can be described completely by their faces, edges, and vertices.
- Some shapes or combinations of shapes can be put together without overlapping to completely cover the plane.
- There is more than one way to classify most shapes and solids.

## BIG IDEA #15

**ORIENTATION & LOCATION: Objects in space can be oriented in an infinite number of ways, and an object's location in space can be described quantitatively.**

**Examples of Mathematical Understandings:**

*Lines and Line Segments*

- Two distinct lines in the plane are either parallel or intersecting; two distinct lines in space are parallel, intersecting or skew.
- The angles formed by two intersecting lines in the plane are related in special ways (e.g., vertical angles).
- A number of degrees can be used to describe the size of an angle's opening.

- Some angles have special relationships based on their position or measures (e.g., complementary angles).
- In the plane, when a line intersects two parallel lines the angles formed are related in special ways.

*Objects*

- The orientation of an object does not change the other attributes of the object.
- The Cartesian Coordinate System is a scheme that uses two perpendicular number lines intersecting at 0 on each to name the location of points in the plane; the system can be extended to name points in space.
- Every point in the plane can be described uniquely by an ordered pair of numbers; the first number tells the distance to the left or right of zero on the horizontal number line; the second tells the distance above or below zero on the vertical number line.

## BIG IDEA #16

### TRANSFORMATIONS: Objects in space can be transformed in an infinite number of ways, and those transformations can be described and analyzed mathematically.

**Examples of Mathematical Understandings:**

- Congruent figures remain congruent through translations, rotations, and reflections.
- Shapes can be transformed to similar shapes (but larger or smaller) with proportional corresponding sides and congruent corresponding angles
- Algebraic expressions can be used to generalize transformations for objects in the plane.
- Some shapes can be divided in half where one half folds exactly on top of the other (line symmetry).
- Some shapes can be rotated around a point in less than one complete turn and land exactly on top of themselves (rotational symmetry).

## BIG IDEA #17

### MEASUREMENT: Some attributes of objects are measurable and can be quantified using unit amounts.

**Examples of Mathematical Understandings:**

- Measurement involves a selected attribute of an object (length, area, mass, volume, capacity) and a comparison of the object being measured against a unit of the same attribute.
- The longer the unit of measure, the fewer units it takes to measure the object.
- The magnitude of the attribute to be measured and the accuracy needed determines the appropriate measurement unit.
- For a given perimeter there can be a shape with area close to zero. The maximum area for a given perimeter and a given number of sides is the regular polygon with that number of sides.

## BIG IDEA #18

### DATA COLLECTION: Some questions can be answered by collecting and analyzing data, and the question to be answered determines the data that needs to be collected and how best to collect it.

**Examples of Mathematical Understandings:**

- An appropriately selected sample can be used to describe and make predictions about a population.
- The size of a sample determines how close data from the sample mirrors the population.

## BIG IDEA #19

**DATA REPRESENTATION: Data can be represented visually using tables, charts, and graphs. The type of data determines the best choice of visual representation.**

**Examples of Mathematical Understandings:**

- Each type of graph is most appropriate for certain types of data.
- Scale influences the patterns that can be observed in data.

## BIG IDEA #20

**DATA DISTRIBUTION: There are special numerical measures that describe the center and spread of numerical data sets.**

**Examples of Mathematical Understandings:**

- The best descriptor of the center of a numerical data set (i.e., mean, median, mode) is determined by the nature of the data and the question to be answered.
- Outliers affect the mean, median, and mode in different ways.
- Data interpretation is enhanced by numerical measures telling how data are distributed.

## BIG IDEA #21

**CHANCE: The chance of an event occurring can be described numerically by a number between 0 and 1 inclusive and used to make predictions about other events.**

**Examples of Mathematical Understandings:**

- Probability can provide a basis for making predictions.
- Some probabilities can only be determined through experimental trials.
- An event that is certain to happen will always happen (The probability is 1.) and an event that is impossible will never happen (The probability is 0.).

APPENDIX A:

# A Sample Chapter Analysis Using Standards and Big Ideas

**GRADE 3 CHAPTER 9: MULTIPLYING GREATER NUMBERS**

## CONTENT STANDARDS

| | |
|---|---|
| **Focus**<br>*NS 2.4: Solve simple problems involving multiplication of multi-digit numbers by one-digit numbers<br><br>**Other**<br>*NS 3.3: Solve problems involving operations with money amounts in decimal notation and multiply and divide money amounts in decimal notation using whole number multipliers and divisors | **Discussion**<br>NS 2.4, Multiplying multi-digit numbers by 1-digit numbers, is the main focus of this chapter. Multiplicands up to four digits are used including money amounts such as $43.98. The standard algorithm is developed with whole numbers and extended to money. All decimal quantities are related to money amounts. |

## BIG IDEAS

| | |
|---|---|
| **Focus**<br>+ Algorithms: Algorithms for operations with rational numbers use notions of equivalence to transform calculations into simpler ones.<br><br>+ Operation Meanings & Relationships: The same number sentence can be associated with different concrete or real-world situations, AND different number sentences can be associated with the same concrete or real-world situation.<br><br>+ Properties: For a given set of numbers there are relationships that are always true, and these are the rules that govern arithmetic and algebra.<br><br>**Other**<br>+ Equivalence: Any number, measure, numerical expression, algebraic expression, or equation can be represented in an infinite number of ways that have the same value. | **Discussion**<br>The Big Idea of focus is Algorithms and it should be emphasized in every skill lesson. All calculations in this chapter involve changing the numerical expression to an equivalent one and breaking the calculation into simpler ones involving basic facts or 1-digit numbers times a multiple of 10, or 100. Lesson 9-1 develops the simpler calculations one needs to know for the other skill lessons in the chapter.<br><br>The array interpretation of multiplication is used to show how the standard algorithm involves breaking the calculation into simpler ones. The distributive property justifies the breaking apart process. For example, 3 x 15 = 3 x (10 + 5) = (3 x 10) + (3 x 5) = 30 + 5 = 35. Notice that 15 is named in an equivalent way, 10 + 5. |

## MATHEMATICAL REASONING

| | |
|---|---|
| **Focus**<br>MR 1.1 Analyze problems by identifying relationships, and so forth<br><br>MR 2.1 Use estimation to verify the reasonableness of results<br><br>MR 2.2 Apply strategies and results from simpler problems to more complex problems.<br><br>**Other**<br>MR 2.3 – 2.6, 3.0 | **Discussion**<br>Estimating products, developed in 9-4, should be emphasized in every lesson thereafter. Rounding might be used most often but compatible numbers also can be used (e.g., 2 x 36 is about 2 x 35). The need for exact or approximate answers should be discussed.<br><br>Operation meanings should be emphasized. The repeated addition and array interpretations are emphasized throughout the chapter.<br><br>The Big Idea of Algorithms connects to MR 2.2, breaking a problem into simpler ones. This applies to the multiplication algorithm and to solving some types of word problems.<br><br>Students should generalize the multiplication process from 2-digit through 4-digit multiplicands; the process is the same, it is just repeated. |

## *References*


Annenberg/CPB. (2004) Learning Math: Five College-Level Mathematics Courses for Elementary and Middle School Teachers. Available at learner.org/channel/courses/learningmath/index as of June 2004.

Clements, D. & DiBiase, A-M. (2001) Guidelines for Early Childhood Mathematics. *Intersection,* Exxon Mobil Foundation and the National Council of Teachers of Mathematics. September 2001 issue. Available at intersectionlive.org/index as of June 2004.

Hiebert, J. & Carpenter, T. P. (1992) Learning and Teaching with Understanding. In *Handbook of Research on Mathematics Teaching and Learning,* edited by Douglas A. Grouws. New York: Macmillan. 65-97.

Hiebert, J., Carpenter, T.P., Fennema, E., Fuson, K., Wearne, D., Murray, H., Olivier, A., & Human, P. (1997). *Making Sense: Teaching and Learning Mathematics with Understanding.* Portsmouth, NH: Heinemann.

Lambdin, D. (2003). Benefits of Teaching through Problem Solving. In *Teaching Mathematics Through Problem Solving: Grades PreK-6,* edited by Frank K. Lester Jr. and Randall I. Charles. Reston, VA: National Council of Teachers of Mathematics. 3-14.

Ma, L. (1999). *Knowing and Teaching Elementary Mathematics: Teachers' Understanding of Fundamental Mathematics in China and the United States.* Mahwah, N.J.: Lawrence Erlbaum.

National Council of Teachers of Mathematics (1969) *Historical Topics for the Mathematics Classroom.* Washington, D.C.: NCTM.

__________ (1989) *Research Issues in the Learning and Teaching of Algebra.* Edited by Sigrid Wagner and Carolyn Kieran. Reston, VA: NCTM.

__________ (1992) *Handbook of Research on Mathematics Teaching and Learning.* Edited by Douglas A. Grouws. New York: Macmillan Publishing.

__________ (2000) *Principles and Standards for School Mathematics.* Reston, VA: NCTM.

National Research Council (1990) *On the Shoulders of Giants: New Approaches to Numeracy.* Edited by Lynn Arthur Steen. Washington, D.C.: National Academy Press.

__________ (2001) *Adding it up: Helping Children Learn Mathematics.* Edited by Jeremy Kilpatrick, Jane Swafford, and Bradford Findell. Washington, D.C.: National Academy Press.

O'Daffer, P., Charles, R., Cooney, T., Dossey, J., and Schielack, J. (2005). *Mathematics for Elementary School Teachers.* 3rd ed. Boston: Addison Wesley.

Ritchhart, R. (1999) Generative Topics: Building a Curriculum Around Big Ideas. *Teaching Children Mathematics* 5. 462-68.

Schifter, D., Russell, S.J., & Bastable, V. (1999) Teaching to the Big Ideas. In *The Diagnostic Teacher: Constructing New Approaches to Professional Development,* edited by Mildred Z. Solomon. New York, New York: Teachers College Press. 22-47.

Southwest Consortium for the Improvement of Mathematics and Science Teaching. (2002) Getting to the Heart of the (Subject) Matter. *Classroom Compass.* September issue. Available at sedl.org/scimast as of June 2004.

Stigler, J. (2004) The Teaching Gap: Reflections on Mathematics Teaching and How to Improve It. Paper presented at the Pearson National Educational Leadership Conference, Washington, D.C., March 2004.

Takahashi, A. & Yoshida, M. (2004) Ideas for Establishing Lesson-Study Communities. *Teaching Children Mathematics* 10. 436-443.

Trafton, P. R. & Reys, B.J. (2004) Standards-Based Mathematics Curriculum Materials: A Phrase in Search of a Definition. *Phi Delta Kappan*. Available at pdkintl.org/kappan/k0111tra.htm as of June 2004.

Van de Walle, J. A. (2001) *Elementary and Middle School Mathematics: Teaching Developmentally.* 4th ed. New York: Longman.

Weiss, I. R., Heck, D.J., & Shimkus, E.S. (2004) Mathematics Teaching in the United States. *Journal of Mathematics Education Leadership* 7. 23-32.

# K–8 Publishers' Criteria for the Common Core State Standards for Mathematics

> These Standards are not intended to be new names for old ways of doing business. They are a call to take the next step. ... It is time to recognize that standards are not just promises to our children, but promises we intend to keep.
>
> –CCSSM, p. 5

The Common Core State Standards were developed through a state-led initiative that drew on the expertise of teachers, researchers and content experts from across the country. The Standards define a staircase to college and career readiness, building on the best of previous state standards and evidence from international comparisons and domestic reports and recommendations. Most states have now adopted the Standards to replace previous expectations in English language arts/literacy and mathematics.

Standards by themselves cannot raise achievement. Standards don't stay up late at night working on lesson plans, or stay after school making sure every student learns—it's teachers who do that. And standards don't implement themselves. Education leaders from the state board to the building principal must make the Standards a reality in schools. Publishers too have a crucial role to play in providing the tools that teachers and students need to meet higher standards. This document, developed by the CCSSM writing team, aims to support faithful CCSSM implementation by providing criteria for materials aligned to the Common Core State Standards for Mathematics.

How should alignment be judged? Traditionally, judging alignment has been approached as a crosswalking exercise. But crosswalking can result in large percentages of "aligned content" while obscuring the fact that the materials in question align not at all to the letter or the spirit of the standards being implemented. These criteria are an attempt to sharpen the alignment question and make alignment and misalignment more clearly visible.

These criteria were developed from the perspective that publishers and purchasers are equally responsible for a healthy materials market. Publishers cannot deliver focus to buyers who only ever complain about what has been left out, yet never complain about what has crept in. More generally, publishers cannot invest in quality if the market doesn't demand it of them nor reward them for producing it.

The document is structured as follows:

I. Focus, Coherence, and Rigor in the Common Core State Standards for Mathematics
II. Criteria for Materials and Tools Aligned to the Standards
III. Appendix: "The Structure is the Standards"

# I. Focus, Coherence, and Rigor in the Common Core State Standards for Mathematics

> Less topic coverage can be associated with higher scores on those topics covered because students have more time to master the content that is taught.
>
> –Ginsburg et al., 2005, *Reassessing U.S. International Mathematics Performance: New Findings from the 2003 TIMSS and PISA*

> This finding that postsecondary instructors target fewer skills as being of high importance is consistent with recent policy statements and findings raising concerns that some states require too many standards to be taught and measured, rather than focusing on the most important state standards for students to attain. ...
>
> Because the postsecondary survey results indicate that a more rigorous treatment of fundamental content knowledge and skills needed for credit-bearing college courses would better prepare students for postsecondary school and work, states would likely benefit from examining their state standards and, where necessary, reducing them to focus only on the knowledge and skills that research shows are essential to college and career readiness and postsecondary success. ...
>
> —*ACT National Curriculum Survey 2009*

> Because the mathematics concepts in [U.S.] textbooks are often weak, the presentation becomes more mechanical than is ideal. We looked at both traditional and non-traditional textbooks used in the U.S. and found conceptual weakness in both.
>
> —Ginsburg et al., 2005, cited in CCSSM, p. 3

> ...[B]ecause conventional textbook coverage is so fractured, unfocused, superficial, and unprioritized, there is no guarantee that most students will come out knowing the essential concepts of algebra.
>
> —Wiggins, 2012[1]

For years national reports have called for greater focus in U.S. mathematics education. TIMSS and other international studies have concluded that mathematics education in the United States is a mile wide and an inch deep. In high-performing countries, strong foundations are laid and then further knowledge is built on them; the design principle in those countries is focus with coherent progressions. The U.S. has lacked such discipline.

There is evidence that state standards have become somewhat more focused over the past decade. But in the absence of standards shared across states, instructional materials have not followed suit. Moreover, prior to the Common Core, state standards were making little progress in terms of coherence: states were not fueling achievement by organizing math so that the subject makes sense.

With the advent of the Common Core, a decade's worth of recommendations for greater focus and coherence finally have a chance to bear fruit. Focus and coherence are the two major evidence-based design principles of the Common Core State Standards for Mathematics. These principles are meant to fuel greater achievement in a rigorous curriculum, in which

[1] From http://grantwiggins.wordpress.com/2012/02/01/a-postscript-to-my-comment-about-kids-having-trouble-with-the-distributive-property.

students acquire conceptual understanding, procedural skill and fluency, and the ability to apply mathematics to solve problems. Thus, the implications of the standards for mathematics education could be summarized briefly as follows:

**Focus:** focus strongly where the standards focus

**Coherence:** think across grades, and link to major topics in each grade

**Rigor:** in major topics, pursue with equal intensity

- conceptual understanding,
- procedural skill and fluency, and
- applications

## Focus

Focus requires that we significantly narrow the scope of content in each grade so that students more deeply experience that which remains.

We have come to see "narrowing" as a bad word—and it is a bad word, if it means cutting arts programs and language programs. But math has swelled in this country. The Standards are telling us that math actually needs to lose a few pounds.

The overwhelming focus of the Standards in early grades is arithmetic along with the components of measurement that support it. That includes the concepts underlying arithmetic, the skills of arithmetic computation, and the ability to apply arithmetic to solve problems and put arithmetic to engaging uses. Arithmetic in the K–5 standards is an important life skill, as well as a thinking subject and a rehearsal for algebra in the middle grades.

Focus remains important through the middle and high school grades in order to prepare students for college and careers; surveys suggest that postsecondary instructors value greater mastery of prerequisites over shallow exposure to a wide array of topics with dubious relevance to postsecondary work.

During the writing of the Standards, the writing team often received feedback along these lines: "I love the focus of these standards! Now, if we could just add one or two more things...." But focus compromised is no longer focus at all. Faithfully implementing the Standards requires moving some topics traditionally taught in earlier grades up to higher grades entirely, sometimes to much higher grades. "Teaching less, learning more" can seem like hard medicine for an educational system addicted to coverage. But remember that the goal of focus is to make good on the ambitious promise the states have made to their students by adopting the Standards: greater achievement at the college- and career-ready level, greater depth of understanding of mathematics, and a rich classroom environment in which reasoning, sense-making, applications, and a range of mathematical practices all thrive. None of this is realistic in a mile-wide, inch-deep world.

Both of the assessment consortia have made the focus, coherence, and rigor of the Standards central to their assessment designs.[2] Choosing materials that also embody the Standards will be essential for giving teachers and students the tools they need to build a strong mathematical foundation and succeed on the coming aligned exams.

## Coherence

Coherence is about making math make sense. Mathematics is not a list of disconnected tricks or mnemonics. It is an elegant subject in which powerful knowledge results from reasoning with a small number of principles such as place value and properties of operations.[3] The standards define progressions of learning that leverage these principles as they build knowledge over the grades.[4]

When people talk about coherence, they often talk about making connections between topics. The most important connections are vertical: the links from one grade to the next that allow students to progress in their mathematical education. That is why it is critical to think across grades and examine the progressions in the standards to see how major content develops over time.

Connections at a single grade level can be used to improve focus, by tightly linking secondary topics to the major work of the grade. For example, in grade 3, bar graphs are not "just another topic to cover." Rather, the standard about bar graphs asks students to use information presented in bar graphs to solve word problems using the four operations of arithmetic. Instead of allowing bar graphs to detract from the focus on arithmetic, the standards are showing how bar graphs can be positioned in support of the major work of the grade. In this way coherence can support focus.

Materials cannot match the contours of the Standards by approaching each individual content standard as a separate event. Nor can materials align to the Standards by approaching each individual grade as a separate event. From the Appendix: "The standards were not so much assembled out of topics as woven out of progressions. Maintaining these progressions in the implementation of the standards will be important for helping all students learn mathematics at a higher level. ... For example, the properties of operations, learned first for simple whole numbers, then in later grades extended to fractions, play a central role in understanding operations with negative numbers, expressions with letters and later still the study of polynomials. As the application of the properties is extended over the grades, an understanding of how the properties of operations work together should deepen and develop into one of the most fundamental insights into algebra. The natural distribution of prior knowledge in classrooms should not prompt abandoning instruction in grade level content, but should prompt explicit attention to connecting grade level content to content from prior learning. To do this, instruction should reflect the progressions on which the CCSSM are built."

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[2] See the Smarter/Balanced content specification and item development specifications, and the PARCC Model Content Framework and item development ITN. Complete information about the consortia can be found at www.smarterbalanced.org and www.parcconline.org.

[3] For some remarks by Phil Daro on this theme, see the excerpt at http://vimeo.com/achievethecore/darofocus, and/or the full video available at http://commoncoretools.me/2012/05/21/phil-daro-on-learning-mathematics-through-problem-solving/.

[4] For more information on progressions in the Standards, see http://ime.math.arizona.edu/progressions.

## Rigor

To help students meet the expectations of the Standards, educators will need to pursue, with equal intensity, three aspects of rigor in the major work of each grade: conceptual understanding, procedural skill and fluency, and applications. The word "understand" is used in the Standards to set explicit expectations for conceptual understanding, the word "fluently" is used to set explicit expectations for fluency, and the phrase "real-world problems" and the star symbol (⋆) is used to set expectations and flag opportunities for applications and modeling (which is a Standard for Mathematical Practice as well as a content category in High School).

To date, curricula have not always been balanced in their approach to these three aspects of rigor. Some curricula stress fluency in computation, without acknowledging the role of conceptual understanding in attaining fluency. Some stress conceptual understanding, without acknowledging that fluency requires separate classroom work of a different nature. Some stress pure mathematics, without acknowledging first of all that applications can be highly motivating for students, and moreover, that a mathematical education should make students fit for more than just their next mathematics course. At another extreme, some curricula focus on applications, without acknowledging that math doesn't teach itself.

The Standards do not take sides in these ways, but rather they set high expectations for all three components of rigor in the major work of each grade. Of course, that makes it necessary that we first follow through on the focus in the Standards—otherwise we are asking teachers and students to do more with less.

# II. Criteria for Materials and Tools Aligned to the Standards

> The single most important flaw in United States mathematics instruction is that the curriculum is "a mile wide and an inch deep." This finding comes from research comparing the U.S. curriculum to high performing countries, surveys of college faculty and teachers, the National Math Panel, the Early Childhood Learning Report, and all the testimony the CCSS writers heard. The standards are meant to be a blueprint for math instruction that is more focused and coherent. ... Crosswalks and alignments and pacing plans and such cannot be allowed to throw away the focus and coherence and regress to the mile-wide curriculum.
>
> —Daro, McCallum, and Zimba, 2012 (from the Appendix)

## Using the criteria

One approach to developing a document such as this one would have been to develop a separate criterion for each mathematical topic approached in deeper ways in the Standards, a separate criterion for each of the Standards for Mathematical Practice, etc. It is indeed necessary for textbooks to align to the Standards in detailed ways. However, enumerating those details here would have led to a very large number of criteria. Instead, the criteria use the Standards' focus, coherence, and rigor as the main themes. In addition, this document includes a section on indicators of quality in materials and tools, as well as a criterion for the mathematics and statistics in instructional resources for science and technical subjects. Note that the criteria apply to materials and tools, not to teachers or teaching.

The criteria can be used in several ways:

- *Informing purchases and adoptions.* Schools or districts evaluating materials and tools for purchase can use the criteria to test claims of alignment. States reviewing materials and tools for adoption can incorporate these criteria into their rubrics. Publishers currently modifying their programs, or designing new materials and tools, can use the criteria to shape these projects.
- *Working with previously purchased materials.* Most existing materials and tools likely fail to meet one or more of these criteria, even in cases where alignment to the Standards is claimed. But the pattern of failure is likely to be informative. States and districts need not wait for "the perfect book" to arrive, but can use the criteria now to carry out a thoughtful plan to modify or combine existing resources in such a way that students' actual learning experiences approach the focus, coherence, and rigor of the Standards. Publishers can develop innovative materials and tools specifically aimed at addressing identified weaknesses of widespread textbooks or programs.
- *Reviewing teacher-developed materials and guiding their development.* Publishers aren't the only source of instructional materials; teachers also create materials and tools, ranging in length from an individual problem set or lesson up to an entire unit or longer. States, districts, schools, and teachers themselves can use the criteria to assess the alignment of teacher-developed materials to the Standards and guide the development of new materials aligned to the Standards.
- *Professional development.* The criteria can be used to support activities that help communicate the shifts in the Standards. For example, teachers can analyze existing materials to reveal how they treat the major work of the grade, or assess how well materials attend to the three aspects of rigor, or determine which problems are key to developing the ideas and skills of the grade.

In all these cases, it is recommended that the criteria for focus be attended to first. By attending first to focus, coherence and rigor may realistically develop. Failing to meet any single focus criterion is enough to show that the materials in question are not aligned to the Standards.

For the sake of brevity, the criteria sometimes refer to parts of the Standards using abbreviations such as 3.MD.7 (an individual content standard), MP.8 (a practice standard), 8.EE.B (a cluster heading), or 4.NBT (a domain heading). Readers of the document should have a copy of the Standards available in order to refer to the indicated text in each case.

These criteria were developed for materials and tools in grades K–8. Some of the criteria may also apply to materials developed for high school courses. Note that an update to this document is planned for early 2013 (it is anticipated that this update will also include high school).

The Standards do not dictate the acceptable forms of instructional resources—to the contrary, they are a historic opportunity to raise student achievement through innovation. Materials and tools of very different forms can meet the criteria that follow, including workbooks, multi-year programs, and targeted interventions. For example, materials and tools that treat a single important topic or domain might be valuable to consider.

This also includes digital or online materials and tools. Digital materials offer substantial promise for conveying mathematics in new and vivid ways and customizing learning. In a digital or online format, diving deeper and reaching back and forth across the grades is easy and often useful. Focus and coherence can be greatly enhanced through dynamic navigation—though, if such capabilities are used poorly, focus and coherence could also be greatly diminished.

As noted in the Standards (p. 4), "All students must have the opportunity to learn and meet the same high standards if they are to access the knowledge and skills necessary in their post-school lives. The Standards should be read as allowing for the widest possible range of students to participate fully from the outset, along with appropriate accommodations to ensure maximum participation of students with special education needs." Thus, **an over-arching criterion** for materials and tools is that they provide supports for special populations such as students with disabilities, English language learners,[5] and gifted students.

## Criteria for Materials and Tools Aligned to the Standards

1. **Focus on Major Work: In any single grade, students and teachers using the materials as designed spend the large majority of their time, approximately three-quarters, on the major work of each grade.** In order to preserve the focus and coherence of the Standards, both assessment consortia have designated clusters as major, additional, or supporting,[6] with clusters designated as major comprising the major work of each grade. Materials are highly unlikely to be aligned to the Standards' focus unless students and teachers using them as designed spend the large majority of their time, approximately three-quarters,[7] on the major work of each grade. In addition, major work should especially predominate in the first half of the year (e.g., in grade 3 this is necessary so that students have sufficient time to build understanding and fluency with multiplication).

   Digital or online materials that allow navigation or have no fixed pacing plan are explicitly designed to ensure that students' time on task meets this criterion.

   Note that an important **subset** of the major work in grades K–8 is the progression that leads toward middle-school algebra (see Table 1, next page). Materials give especially careful treatment to these clusters and their interconnections.

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[5] Slides from a brief and informal presentation by Phil Daro about mathematical language and English language learners can be found at http://db.tt/VARV3ebl.

[6] For cluster-level emphases at grades K–2, see http://www.achievethecore.org/downloads/Math%20Shifts%20and%20Major%20Work%20of%20Grade.pdf.

[7] Given the particular clusters that are designated major in grade 7, the criterion for that grade is approximately two-thirds, rather than approximately three-fourths.

Table 1. Progress to Algebra in Grades K–8

| K | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| Know number names and the count sequence | Represent and solve problems involving addition and subtraction | Represent and solve problems involving addition and subtraction | Represent & solve problems involving multiplication and division | Use the four operations with whole numbers to solve problems | Understand the place value system | Apply and extend previous understandings of multiplication and division to divide fractions by fractions | Apply and extend previous understanding of operations with fractions to add, subtract, multiply, and divide rational numbers | Work with radical and integer exponents |
| Count to tell the number of objects | Understand and apply properties of operations and the relationship between addition and subtraction | Add and subtract within 20 | Understand properties of multiplication and the relationship between multiplication and division | Generalize place value understanding for multi-digit whole numbers | Perform operations with multi-digit whole numbers and decimals to hundredths | Apply and extend previous understandings of numbers to the system of rational numbers | Analyze proportional relationships and use them to solve real-world and mathematical problems | Understand the connections between proportional relationships, lines, and linear equations |
| Compare numbers | Add and subtract within 20 | Understand place value | Multiply & divide within 100 | Use place value understanding and properties of operations to perform multi-digit arithmetic | Use equivalent fractions as a strategy to add and subtract fractions | Understand ratio concepts and use ratio reasoning to solve problems | Use properties of operations to generate equivalent expressions | Analyze and solve linear equations and pairs of simultaneous linear equations |
| Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from | Work with addition and subtraction equations | Use place value understanding and properties of operations to add and subtract | Solve problems involving the four operations, and identify & explain patterns in arithmetic | Extend understanding of fraction equivalence and ordering | Apply and extend previous understandings of multiplication and division to multiply and divide fractions | Apply and extend previous understandings of arithmetic to algebraic expressions | Solve real-life and mathematical problems using numerical and algebraic expressions and equations | Define, evaluate, and compare functions |
| Work with numbers 11-19 to gain foundations for place value | Extend the counting sequence | Measure and estimate lengths in standard units | Develop understanding of fractions as numbers | Build fractions from unit fractions by applying and extending previous understandings of operations | Geometric measurement: understand concepts of volume and relate volume to multiplication and to addition | Reason about and solve one-variable equations and inequalities | | Use functions to model relationships between quantities* |
| | Understand place value | Relate addition and subtraction to length | Solve problems involving measurement and estimation of intervals of time, liquid volumes, & masses of objects | Understand decimal notation for fractions, and compare decimal fractions | Graph points in the coordinate plane to solve real-world and mathematical problems* | Represent and analyze quantitative relationships between dependent and independent variables | | |
| | Use place value understanding and properties of operations to add and subtract | | Geometric measurement: understand concepts of area and relate area to multiplication and to addition | | | | | |
| | Measure lengths indirectly and by iterating length units | | | | | | | |

*Indicates a cluster that is well thought of as part of a student's progress to algebra, but that is currently not designated as Major by one or both of the assessment consortia in their draft materials. Apart from the two asterisked exceptions, the clusters listed here are a subset of those designated as Major in both of the assessment consortia's draft documents.

2. **Focus in Early Grades: Materials do not assess any of the following topics before the grade level indicated.**

Table 2

| Topic | Grade Introduced in the Standards |
|---|---|
| **Probability**, including chance, likely outcomes, probability models. | 7 |
| **Statistical distributions**, including center, variation, clumping, outliers, mean, median, mode, range, quartiles; and **statistical association or trends**, including two-way tables, bivariate measurement data, scatter plots, trend line, line of best fit, correlation. | 6 |
| **Similarity, congruence, or geometric transformations.** | 8 |
| **Symmetry** of shapes, including line/reflection symmetry, rotational symmetry. | 4 |

Additionally, materials do not assess pattern problems in K–5 that do not support the focus on arithmetic, such as "find the next one" problems.

As Table 2 indicates, the Standards as a whole do include these topics—they are not being left out. However, in the coherent progression of the Standards, these topics first appear at later grades in order to establish focus. Thus, in aligned materials there are no chapter tests, unit tests, or other assessment components that make students or teachers responsible for any of the above topics before the grade in which they are introduced in the Standards. (One way to meet this criterion is for materials to omit these topics entirely prior to the indicated grades.)

3. **Focus and Coherence through Supporting Work: Supporting content does not detract from focus, but rather enhances focus and coherence simultaneously by engaging students in the major work of the grade.** For example, materials for K–5 generally treat data displays as an occasion for solving grade-level word problems using the four operations.[8] (This criterion does not apply in the case of targeted supplemental materials or other tools that do not include supporting content.)

4. **Rigor and Balance: Materials and tools reflect the balances in the Standards and help students meet the Standards' rigorous expectations, by (all of the following, in the case of comprehensive materials; at least one of the following for supplemental or targeted resources):**

   a. **Developing students' <u>conceptual understanding</u> of key mathematical concepts, where called for in specific content standards or cluster headings.** Materials amply feature

[8] For more information about this example, see Table 1 in the *Progression* for K-3 Categorical Data and 2-5 Measurement Data, http://commoncoretools.files.wordpress.com/2011/06/ccss_progression_md_k5_2011_06_20.pdf. More generally, the *PARCC Model Content Frameworks* give examples in each grade of how to improve focus and coherence by linking supporting topics to the major work.

high-quality conceptual problems and questions that can serve as fertile conversation-starters in a classroom if students are unable to answer them. This includes brief conceptual problems with low computational difficulty (e.g., 'Find a number greater than 1/5 and less than 1/4'); brief conceptual questions (e.g., 'If the divisor does not change and the dividend increases, what happens to the quotient?'); and problems that involve identifying correspondences across different mathematical representations of quantitative relationships.[9] In the materials, conceptual understanding is not a generalized imperative applied with a broad brush, but is attended to most thoroughly in those places in the content standards where explicit expectations are set for understanding or interpreting. Such problems and activities include fine-grained mathematical concepts, such as place value, the whole-number product $a \times b$, the fraction $^a/_b$, the fraction product $(^a/_b) \times q$, expressions as records of calculations, solving equations as a process of answering a question, etc. (Conceptual understanding of key mathematical concepts is thus distinct from applications or fluency work, and these three aspects of rigor must be balanced as indicated in the Standards.)

b. **Giving attention throughout the year to individual standards that set an expectation of <u>fluency</u>**. The Standards are explicit where fluency is expected. Materials in grades K–6 help students make steady progress throughout the year toward fluent (accurate and reasonably fast) computation, including knowing single-digit products and sums from memory (see, e.g., 2.OA.2 and 3.OA.7). Progress toward these goals is interwoven with students' developing conceptual understanding of the operations in question.[10] Manipulatives and concrete representations such as diagrams that enhance conceptual understanding are closely connected to the written and symbolic methods to which they refer (see, e.g., 1.NBT). As well, purely procedural problems and exercises are present. These include cases in which opportunistic strategies are valuable—e.g., the sum 698 + 240 or the system $x + y = 1$, $2x + 2y = 3$—as well as an ample number of generic cases so that students can learn and practice efficient algorithms (e.g., the sum 8767 + 2286). Methods and algorithms are general and based on principles of mathematics, not mnemonics or tricks.[11] Materials do not make fluency a generalized imperative to be applied with a broad brush, but attend most thoroughly to those places in the content standards where explicit expectations are set for fluency. In higher grades, algebra is the language of much of mathematics. Like learning any language, we learn by using it. Sufficient practice with algebraic operations is provided so as to make realistic the attainment of the Standards as a whole; for example, fluency in algebra can

[9] Note that for ELL students, multiple representations also serve as multiple access paths.

[10] For more about how students develop fluency in tandem with understanding, see the *Progressions* for Operations and Algebraic Thinking, http://commoncoretools.files.wordpress.com/2011/05/ccss_progression_cc_oa_k5_2011_05_302.pdf and for Number and Operations in Base Ten, http://commoncoretools.files.wordpress.com/2011/04/ccss_progression_nbt_2011_04_073.pdf.

[11] Non-mathematical approaches (such as the "butterfly method" of adding fractions) compromise focus and coherence and displace mathematics in the curriculum (cf. 5.NF.1). For additional background on this point, see the remarks by Phil Daro excerpted at http://vimeo.com/achievethecore/darofocus and/or the full video, available at http://commoncoretools.me/2012/05/21/phil-daro-on-learning-mathematics-through-problem-solving/.

help students get past the need to manage computational details so that they can observe structure (MP.7) and express regularity in repeated reasoning (MP.8).

c. **Allowing teachers and students using the materials as designed to spend sufficient time working with engaging applications, without losing focus on the major work of each grade.** Materials in grades K–8 include an ample number of single-step and multi-step contextual problems that develop the mathematics of the grade, afford opportunities for practice, and engage students in problem solving. Materials for grades 6–8 also include problems in which students must make their own assumptions or simplifications in order to model a situation mathematically. Applications take the form of problems to be worked on individually as well as classroom activities centered on application scenarios. Materials attend thoroughly to those places in the content standards where expectations for multi-step and real-world problems are explicit. Applications in the materials draw only on content knowledge and skills specified in the content standards, with particular stress on applying major work, and a preference for the more fundamental techniques from additional and supporting work. Modeling builds slowly across K–8, and applications are relatively simple in earlier grades. Problems and activities are grade-level appropriate, with a sensible tradeoff between the sophistication of the problem and the difficulty or newness of the content knowledge the student is expected to bring to bear.[12]

**Additional aspects of the Rigor and Balance Criterion:**
(1) *The three aspects of rigor are not always separate in materials.* (Conceptual understanding needs to underpin fluency work; fluency can be practiced in the context of applications; and applications can build conceptual understanding.)

(2) *Nor are the three aspects of rigor always together in materials.* (Fluency requires dedicated practice to that end. Rich applications cannot always be shoehorned into the mathematical topic of the day. And conceptual understanding will not come along for free unless explicitly taught.)

(3) Digital and online materials with no fixed lesson flow or pacing plan are not designed for superficial browsing but rather instantiate the Rigor and Balance criterion and promote depth and mastery.

5. **Consistent Progressions: Materials are consistent with the progressions in the Standards, by (all of the following):**

a. **Basing content progressions on the grade-by-grade progressions in the Standards.** Progressions in materials match closely with those in the Standards. This does not require the table of contents in a book to be a replica of the content standards; but the match between the Standards and what students are to learn should be close in each grade. Discrepancies are clearly aimed at helping students meet the Standards as

[12] Cf. CCSSM, p. 84. Also note that modeling is a mathematical practice in every grade, but in high school it is also a content category (CCSSM, pp. 72, 73); therefore, modeling is generally enhanced in high school materials, with more elements of the modeling cycle (CCSSM, p. 72).

written, rather than effectively rewriting the standards. Comprehensive materials do not introduce gaps in learning by omitting content that is specified in the Standards.

The basic model for grade-to-grade progression involves students making tangible progress during each given grade, as opposed to substantially reviewing then marginally extending from previous grades. Grade-level work begins during the first two to four weeks of instruction, rather than being deferred until later as previous years' content is reviewed. Remediation may be necessary, particularly during transition years, and resources for remediation may be provided, but review is clearly identified as such to the teacher, and teachers and students can see what their specific responsibility is for the current year.

Digital and online materials that allow students and/or teachers to navigate content across grade levels promote the Standards' coherence by tracking the structure and progressions in the Standards. For example, such materials might link problems and concepts so that teachers and students can browse a progression.

b. **Giving all students extensive work with grade-level problems.** Differentiation is sometimes necessary, but materials often manage unfinished learning from earlier grades inside grade-level work, rather than setting aside grade-level work to reteach earlier content. Unfinished learning from earlier grades is normal and prevalent; it should not be ignored nor used as an excuse for cancelling grade level work and retreating to below-grade work. (For example, the development of fluency with division using the standard algorithm in grade 6 is the occasion to surface and deal with unfinished learning about place value; this is more productive than setting aside division and backing up.) Likewise, students who are "ready for more" can be provided with problems that take grade-level work in deeper directions, not just exposed to later grades' topics.

c. **Relating grade level concepts explicitly to prior knowledge from earlier grades.** The materials are designed so that prior knowledge becomes reorganized and extended to accommodate the new knowledge. Grade-level problems in the materials often involve application of knowledge learned in earlier grades. Although students may well have learned this earlier content, they have not learned how it extends to new mathematical situations and applications. They learn basic ideas of place value, for example, and then extend them across the decimal point to tenths and beyond. They learn properties of operations with whole numbers, and then extend them to fractions, variables, and expressions. The materials make these extensions of prior knowledge explicit. Note that cluster headings in the Standards sometimes signal key moments where reorganizing and extending previous knowledge is important in order to accommodate new knowledge (e.g., see the cluster headings that use the phrase "Apply and extend previous understanding").

6. **Coherent Connections: Materials foster coherence through connections at a single grade, where appropriate and where required by the Standards, by (all of the following):**

a. **Including learning objectives that are visibly shaped by CCSSM cluster headings, with meaningful consequences for the associated problems and activities.** While some clusters are simply the sum of their individual standards (e.g., 8.EE.C), many are not (e.g., 8.EE.B). In the latter cases, cluster headings function like topic sentences in a paragraph in that they state the point of, and lend additional meaning to, the individual content standards that follow. Cluster headings can also signal multi-grade progressions, by using phrases such as "Apply and extend previous understandings of [X] to do [Y]." Hence an important criterion for coherence is that some or many of the learning objectives in the materials are visibly shaped by CCSSM cluster headings, with meaningful consequences for the associated problems and activities. Materials do not simply treat the Standards as a sum of individual content standards and individual practice standards.

b. **Including problems and activities that serve to connect two or more clusters in a domain, or two or more domains in a grade, in cases where these connections are natural and important.** If instruction only operates at the individual standard level, or even at the individual cluster level, then some important connections will be missed. For example, robust work in 4.NBT should sometimes or often synthesize across the clusters listed in that domain; robust work in grade 4 should sometimes or often involve students applying their developing computation NBT skills in the context of solving word problems detailed in OA. Materials do not invent connections not explicit in the standards without first attending thoroughly to the connections that are required explicitly in the Standards (e.g., 3.MD.7 connects area to multiplication, to addition, and to properties of operations; A-REI.11 connects functions to equations in a graphical context.) Not everything in the standards is naturally well connected or needs to be connected (e.g., Order of Operations has essentially nothing to do with the properties of operations, and connecting these two things in a lesson or unit title is actively misleading). Instead, connections in materials are mathematically natural and important (e.g., base-ten computation in the context of word problems with the four operations), reflecting plausible direct implications of what is written in the Standards without creating additional requirements.

7. **Practice-Content Connections: Materials meaningfully connect content standards and practice standards.** "Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction." (CCSSM, p. 8.) Over the course of any given year of instruction, each mathematical practice standard is meaningfully present in the form of activities or problems that stimulate students to develop the habits of mind described in the practice standards. These practices are well-grounded in the content standards. Materials are accompanied by an analysis, aimed at evaluators, of how the authors have approached each practice standard in relation to content within each applicable grade or grade band. Materials do not treat the practice standards as static across grades or grade bands, but instead tailor the connections to the content of the grade and to grade-level-appropriate student thinking. Materials also include teacher-directed materials that explain the role of the practice standards in the classroom and in students' mathematical development.

8. **Focus and Coherence via Practice Standards: Materials promote focus and coherence by connecting practice standards with content that is emphasized in the Standards.** Content and practice standards are not connected mechanistically or randomly, but instead support focus and coherence. Examples: Materials connect looking for and making use of structure (MP.7) with structural themes emphasized in the Standards such as properties of operations, place value decompositions of numbers, numerators and denominators of fractions, numerical and algebraic expressions, etc; materials connect looking for and expressing regularity in repeated reasoning (MP.8) with major topics by using regularity in repetitive reasoning as a *tool* with which to explore major topics. (In K–5, materials might use regularity in repetitive reasoning to shed light on, e.g., the $10 \times 10$ addition table, the $10 \times 10$ multiplication table, the properties of operations, the relationship between addition and subtraction or multiplication and division, and the place value system; in 6–8, materials might use regularity in repetitive reasoning to shed light on proportional relationships and linear functions; in high school, materials might use regularity in repetitive reasoning to shed light on formal algebra as well as functions, particularly recursive definitions of functions.)

9. **Careful Attention to Each Practice Standard: Materials attend to the full meaning of each practice standard.** For example, MP.1 does not say, "Solve problems." Or "Make sense of problems." Or "Make sense of problems and solve them." It says "Make sense of problems and persevere in solving them." Thus, students using the materials as designed build their perseverance in grade-level-appropriate ways by occasionally solving problems that require them to persevere to a solution beyond the point when they would like to give up. MP.5 does not say, "Use tools." Or "Use appropriate tools." It says "Use appropriate tools strategically." Thus, materials include problems that reward students' strategic decisions about how to use tools, or about whether to use them at all. MP.8 does not say, "Extend patterns." Or "Engage in repetitive reasoning." It says "Look for and express regularity in repeated reasoning." Thus, it is not enough for students to extend patterns or perform repeated calculations. Those repeated calculations must lead to an insight (e.g., "When I add a multiple of 3 to another multiple of 3, then I get a multiple of 3."). The analysis for evaluators explains how the full meaning of each practice standard has been attended to in the materials.

10. **Emphasis on Mathematical Reasoning: Materials support the Standards' emphasis on mathematical reasoning, by (all of the following):**

    a. **Prompting students to construct viable arguments and critique the arguments of others concerning key grade-level mathematics that is detailed in the content standards (cf. MP.3).** Materials provide sufficient opportunities for students to reason mathematically in independent thinking and express reasoning through classroom discussion and written work. Reasoning is not confined to optional or avoidable sections of the materials but is inevitable when using the materials as designed. Materials do not approach reasoning as a generalized imperative, but instead create opportunities for students to reason *about* key mathematics detailed in the content standards for the grade. Materials thus attend first and most thoroughly to those places in the content standards setting explicit expectations for explaining, justifying, showing, or proving. Students are asked to critique given arguments, e.g., by explaining under what conditions, if any, a mathematical statement is valid. Materials develop students' capacity for mathematical reasoning in a grade-level appropriate way, with a reasonable progression of sophistication from early grades up through high school.[13] Teachers and students using the materials as designed spend from a quarter to a half of their classroom time communicating reasoning (by constructing viable arguments and explanations and critiquing those of others' concerning key grade-level mathematics)—recognizing that learning mathematics also involves time spent working on applications and practicing procedures. Materials provide examples of student explanations and arguments (e.g., fictitious student characters might be portrayed).

b. **Engaging students in problem solving as a form of argument**. Materials attend thoroughly to those places in the content standards that explicitly set expectations for multi-step problems; multi-step problems are not scarce in the materials. Some or many of these problems require students to devise a strategy autonomously. Sometimes the goal is the final answer alone (cf. MP.1); sometimes the goal is to show work and lay out the solution as a sequence of well justified steps. In the latter case, the solution to a problem takes the form of a cogent argument that can be verified and critiqued, instead of a jumble of disconnected steps with a scribbled answer indicated by drawing a circle around it (cf. MP.6). Problems and activities of this nature are grade-level appropriate, with a reasonable progression of sophistication from early grades up through high school.

c. **Explicitly attending to the specialized language of mathematics.** Mathematical reasoning involves specialized language. Therefore, materials and tools address the development of mathematical and academic language associated with the standards. The language of argument, problem solving and mathematical explanations are taught rather than assumed. Correspondences between language and multiple mathematical representations including diagrams, tables, graphs, and symbolic expressions are identified in material designed for language development. Note that variety in formats and types of representations—graphs, drawings, images, and tables in addition to text—can relieve some of the language demands that English language learners face when they have to show understanding in math.

   The text is considerate of English language learners, helping them to access challenging mathematics and helping them to develop grade level language. For example, materials might include annotations to help with comprehension of words, sentences and paragraphs, and give examples of the use of words in other situations. Modifications to language do not sacrifice the mathematics, nor do they put off necessary language development.

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[13] As students progress through the grades, their production and comprehension of mathematical arguments evolves from informal and concrete toward more formal and abstract. In early grades students employ imprecise expressions which with practice over time become more precise and viable arguments in later grades. Indeed, the use of imprecise language is part of the process in learning how to make more precise arguments in mathematics. Ultimately, conversation about arguments helps students transform assumptions into explicit and precise claims.

A criterion for the mathematics and statistics in materials for science and technical subjects

Lack of alignment between mathematics and science or technical subjects could have the effect of compromising the focus and coherence of the mathematics Standards. Instead of reinforcing concepts and skills already carefully introduced in math class, teachers of science and technical subjects would have to teach this material in stopgap fashion. That wouldn't serve students well in any grade, and elementary teachers in particular would preside over a chaotic learning environment.

[S] **Consistency with CCSSM: Materials for science and technical subjects are consistent with CCSSM.** Materials for these subjects in K–8 do not subtract from the focus and coherence of the Standards by outpacing CCSSM math or data progressions in grades K–8 or misaligning to them. In grades 6–8 and high school, materials for these subjects also build coherence across the curriculum and support college and career readiness by integrating key mathematics into the disciplines, particularly simple algebra in the physical sciences and technical subjects, and basic statistics in the life sciences and technical subjects (see Table 3).

Table 3

| **Algebraic competencies integrated into materials for middle school and high school science and technical subjects** | **Statistical competencies integrated into materials for middle school and high school science and technical subjects** |
|---|---|
| • Working with positive and negative numbers (including fractions) to solve problems<br>• Using variables and writing and solving equations to solve problems<br>• Recognizing and using proportional relationships to solve problems<br>• Graphing proportional relationships and linear functions to solve problems | • Working with distributions and measures of center and variability<br>• Working with simple probability and random sampling<br>• Working with bivariate categorical data (e.g., two-way tables)<br>• Working with bivariate measurement data (e.g., scatter plots) and linear models |

Indicators of quality in instructional materials and tools for mathematics

The preceding criteria express important dimensions of alignment to the Standards. The following are some additional dimensions of quality that materials and tools should exhibit in order to give teachers and students the tools they need to meet the Standards:

- Problems in the materials are worth doing:
  - The underlying design of the materials distinguishes between *problems* and *exercises*. Whatever specific terms are used for these two types, in essence the difference is that in solving problems, students learn new mathematics, whereas in working exercises, students apply what they have already learned to build mastery. Problems are problems because students haven't yet learned how to solve them; students are learning from solving them. Materials use problems to teach mathematics. Lessons have a few well designed problems that progressively build and extend understanding. Practice exercises that build fluency are easy to recognize for their purpose. Other exercises require longer chains of reasoning.
  - Each problem or exercise has a purpose—whether to teach new knowledge, bring misconceptions to the surface, build skill or fluency, engage the student in one or several mathematical practices, or simply present the student with a fun puzzle.
  - Assignments aren't haphazardly designed. Exercises are given to students in intentional sequences—for example, a sequence leading from prior knowledge to new knowledge, or a sequence leading from concrete to abstract, or a sequence that leads students through a number of important cases, or a sequence that elicits new understanding by inviting students to see regularity in repeated reasoning. Lessons with too many problems make problems a commodity; they forbid concentration, and they make focus and coherence unlikely.
  - The language in which problems are posed is carefully considered. Note that mathematical problems posed using only ordinary language are a special genre of text that has conventions and structures needing to be learned. The language used to pose mathematical problems should evolve with the grade level and across mathematics content.

- There is variety in what students produce: Students are assigned to produce answers and solutions, but also arguments and explanations, diagrams, mathematical models, etc.

- There is variety in the pacing and grain size of content coverage.
  - Materials that devote roughly equal time to each content standard do not allow teachers and students to focus where necessary.
  - The Standards are not written at uniform grain size. Sometimes an individual content standard will require days of work, while other standards will be sufficiently addressed when grouped with other standards. For example, it isn't plausible that students will understand concepts of place value (e.g., 2.NBT.1) without substantial explicit instruction, problem solving, and exercises devoted to this particular point.

- There are separate teacher materials that support and reward teacher study, including:
  - Discussion of student ways of thinking with respect to important mathematical problems and concepts—especially anticipating the variety of student responses.
  - Guidance on interaction with students, mostly questions to prompt ways of thinking.
  - Guidance on lesson flow.
  - Discussion of desired mathematical behaviors being elicited among the students.

- The use of manipulatives follows best practices (see, e.g., *Adding It Up*, 2001):
  - *Manipulatives are faithful representations of the mathematical objects they represent.* For example, colored chips can be helpful in representing some features of rational numbers, but they do not provide particularly direct representations of all of the important mathematics. The opposite of the opposite of red isn't clearly blue, for example, and chips aren't particularly well suited as models for adding rational numbers that are not integers (for this, a number line model may be more appropriate).
  - *Manipulatives are closely connected to written methods.* "Research indicates that students' experiences using physical models to represent hundreds, tens, and ones can be effective *if* the materials help them think about how to combine quantities and, eventually, how these processes connect with written procedures." (*Adding It Up*, p. 198, emphasis in the original). For example, base-ten blocks are a reasonable *model* for adding within 1000, but not a reasonable *method* for doing so; nor are colored chips a reasonable *method* for adding integers. (Cf. standards 1.NBT.4, 1.NBT.6, 2.NBT.7, and 5.NBT.7; these are not the only places in the curriculum where connecting to a written method is important). The word "fluently" in particular as used in the Standards refers to fluency with a written or mental method, not a method using manipulatives or concrete representations.

- Materials are carefully reviewed by qualified individuals, whose names are listed, to ensure:
  - Freedom from mathematical errors[14]
  - Grade-level appropriateness
  - Freedom from bias (for example, problem contexts that use culture-specific background knowledge do not assume readers from all cultures have that knowledge; simple explanations or illustrations or hints scaffold comprehension).
  - Freedom from unnecessary language complexity.

- The visual design isn't distracting or chaotic, or aimed at adult purchasers, but instead serves only to support young students in engaging thoughtfully with the subject.

[14] Sometimes errors in materials are simple falsehoods, e.g., printing an incorrect answer to a problem; other errors are more subtle, e.g., asking students to explain why something is so when it has been defined to be so.

- Support for English language learners and members of other special populations is thoughtful and helps those learners to meet the same standards as all other students. Allowing English language learners to collaborate as they strive to learn and show understanding in an environment where English is used as the medium of instruction will give them the support they need to meet their academic goals. Materials can structure interactions in pairs, in small groups, and in the large group (or in any other group configuration), as some English language learners might be shy to share orally with the large group, but might not have problem sharing orally with a small group or in pairs. (In addition, when working in pairs, if English language leaners are paired up with a student who shares the same language, they might choose to think about and discuss the problems in their first language, and then worry about doing it in English.)

- (For paper-based materials.) A textbook that is focused is short. For example, by design Japanese textbooks have less than one page per lesson. Elementary textbooks should be less than 200 pages, middle and secondary less than 500 pages.

Appendix

## The Structure is the Standards

*Essay by Phil Daro, William McCallum, and Jason Zimba, February 16, 2012*[15]

You have just purchased an expensive Grecian urn and asked the dealer to ship it to your house. He picks up a hammer, shatters it into pieces, and explains that he will send one piece a day in an envelope for the next year. You object; he says "don't worry, I'll make sure that you get every single piece, and the markings are clear, so you'll be able to glue them all back together. I've got it covered." Absurd, no? But this is the way many school systems require teachers to deliver mathematics to their students; one piece (i.e. one standard) at a time. They promise their customers (the taxpayers) that by the end of the year they will have "covered" the standards.

In the Common Core State Standards, individual statements of what students are expected to understand and be able to do are embedded within domain headings and cluster headings designed to convey the structure of the subject. "The Standards" refers to all elements of the design—the wording of domain headings, cluster headings, and individual statements; the text of the grade level introductions and high school category descriptions; the placement of the standards for mathematical practice at each grade level.

The pieces are designed to fit together, and the standards document fits them together, presenting a coherent whole where the connections within grades and the flows of ideas across grades are as visible as the story depicted on the urn.

The analogy with the urn only goes so far; the Standards are a policy document, after all, not a work of art. In common with the urn, however, the Standards were crafted to reward study on multiple levels: from close inspection of details, to a coherent grasp of the whole. Specific phrases in specific standards are worth study and can carry important meaning; yet this meaning is also importantly shaped by the cluster heading in which the standard is found. At higher levels, domain headings give structure to the subject matter of the discipline, and the practices' yearly refrain communicates the varieties of expertise which study of the discipline develops in an educated person.

Fragmenting the Standards into individual standards, or individual bits of standards, erases all these relationships and produces a sum of parts that is decidedly less than the whole. Arranging the Standards into new categories also breaks their structure. It constitutes a remixing of the Standards. There is meaning in the cluster headings and domain names that is not contained in the numbered statements beneath them. Remove or reword those headings and you have changed the meaning of the Standards; you now have different Standards; you have not adopted the Common Core.

Sometimes a remix is as good as or better than the original. Maybe there are 50 remixes, adapted to the preferences of each individual state (although we doubt there are 50 good ones). Be that as it may, a remix of a work is not the same as the original work, and with 50 remixes we would not have common standards; we would have the same situation we had before the Common Core.

[15] http://commoncoretools.me/2012/02/16/the-structure-is-the-standards/.

Why is paying attention to the structure important? Here is why: The single most important flaw in United States mathematics instruction is that the curriculum is "a mile wide and an inch deep." This finding comes from research comparing the U.S. curriculum to high performing countries, surveys of college faculty and teachers, the National Math Panel, the Early Childhood Learning Report, and all the testimony the CCSS writers heard. The standards are meant to be a blueprint for math instruction that is more focused and coherent. The focus and coherence in this blueprint is largely in the way the standards progress from each other, coordinate with each other and most importantly cluster together into coherent bodies of knowledge. Crosswalks and alignments and pacing plans and such cannot be allowed to throw away the focus and coherence and regress to the mile-wide curriculum.

Another consequence of fragmenting the Standards is that it obscures the progressions in the standards. The standards were not so much assembled out of topics as woven out of progressions. Maintaining these progressions in the implementation of the standards will be important for helping all students learn mathematics at a higher level. Standards are a bit like the growth chart in a doctor's office: they provide a reference point, but no child follows the chart exactly. By the same token, standards provide a chart against which to measure growth in children's knowledge. Just as the growth chart moves ever upward, so standards are written as though students learned 100% of prior standards. In fact, all classrooms exhibit a wide variety of prior learning each day. For example, the properties of operations, learned first for simple whole numbers, then in later grades extended to fractions, play a central role in understanding operations with negative numbers, expressions with letters and later still the study of polynomials. As the application of the properties is extended over the grades, an understanding of how the properties of operations work together should deepen and develop into one of the most fundamental insights into algebra. The natural distribution of prior knowledge in classrooms should not prompt abandoning instruction in grade level content, but should prompt explicit attention to connecting grade level content to content from prior learning. To do this, instruction should reflect the progressions on which the CCSSM are built. For example, the development of fluency with division using the standard algorithm in grade 6 is the occasion to surface and deal with unfinished learning with respect to place value. Much unfinished learning from earlier grades can be managed best inside grade level work when the progressions are used to understand student thinking.

This is a basic condition of teaching and should not be ignored in the name of standards. Nearly every student has more to learn about the mathematics referenced by standards from earlier grades. Indeed, it is the nature of mathematics that much new learning is about extending knowledge from prior learning to new situations. For this reason, teachers need to understand the progressions in the standards so they can see where individual students and groups of students are coming from, and where they are heading. But progressions disappear when standards are torn out of context and taught as isolated events.

*Sample Rubric. (In each case, the top-line criterion is shown. Refer to the additional text to inform judgment on each criterion.)*

| | Top-Line Criterion | Notes | Evaluation (check one) | |
|---|---|---|---|---|
| **1. Focus on Major Work** | In any single grade, students and teachers using the materials as designed spend the large majority of their time, approximately three-quarters,[16] on the major work of each grade. | | Not Met ☐ Met ☐ | |
| **2. Focus in Early Grades** | Materials do not assess any of the topics in Table 2 before the grade level indicated, or pattern problems in K–5 that do not support the focus on arithmetic, such as "find the next one" problems. | | Not Met ☐ Met ☐ | |
| **3. Focus and Coherence through Supporting Work** | Supporting content (where present) does not detract from focus, but rather enhances focus and coherence simultaneously by engaging students in the major work of the grade. | | Not Met ☐ Met ☐ | |
| **4. Rigor and Balance** Materials and tools reflect the balances in the Standards and help students meet the Standards' rigorous expectations, by (all of the following, in the case of comprehensive materials; at least one of the following for supplemental or targeted resources): | Developing students' <u>conceptual understanding</u> of key mathematical concepts, where called for in specific content standards or cluster headings. | | Not Met ☐ Met ☐ | Comprehensive programs meet all three; supplemental resources meet one or more. |
| | Giving attention throughout the year to individual standards that set an expectation of <u>fluency</u>. | | Not Met ☐ Met ☐ | |
| | Allowing teachers and students using the materials as designed to spend sufficient time working with engaging <u>applications</u>, without losing focus on the major work of each grade. | | Not Met ☐ Met ☐ | |
| **Additional aspects** of the Rigor and Balance criterion | (The three aspects of rigor—if all were checked above—are not always together, not always apart; digital tools are designed to support the rigor and balance criterion and promote depth and mastery.) | | Not Met ☐ Met ☐ | |

[16] Given the particular clusters that are designated major in grade 7, the criterion for that grade is approximately two-thirds, rather than approximately three-fourths.

*Sample Rubric. (In each case, the top-line criterion is shown. Refer to the additional text to inform judgment on each criterion.)*

| | Top-Line Criterion | Notes | Evaluation (check one) |
|---|---|---|---|
| **5. Consistent Progressions**<br>Materials are consistent with the progressions in the Standards, by (all of the following): | Basing content progressions on the grade-by-grade progressions in the Standards. | | Not Met ☐ Met ☐ |
| | Giving all students extensive work with grade-level problems. | | Not Met ☐ Met ☐ |
| | Relating grade level concepts explicitly to prior knowledge from earlier grades. | | Not Met ☐ Met ☐ |
| **6. Coherent Connections**<br>Materials foster coherence through connections at a single grade, where appropriate and where required by the Standards, by (all of the following): | Including learning objectives that are visibly shaped by CCSSM cluster headings, with meaningful consequences for the associated problems and activities. | | Not Met ☐ Met ☐ |
| | Including problems and activities that serve to connect two or more clusters in a domain, or two or more domains in a grade, in cases where these connections are natural and important. | | Not Met ☐ Met ☐ |
| **7. Practice-Content Connections** | Materials meaningfully connect content standards and practice standards. | | Not Met ☐ Met ☐ |

*Sample Rubric. (In each case, the top-line criterion is shown. Refer to the additional text to inform judgment on each criterion.)*

| | Top-Line Criterion | Notes | Evaluation (check one) |
|---|---|---|---|
| **8. Focus and Coherence via Practice Standards** | Materials promote focus and coherence by connecting practice standards with content that is emphasized in the Standards. | | Not Met ☐ Met ☐ |
| **9. Careful Attention to Each Practice Standard** | Materials attend to the full meaning of each practice standard. | | Not Met ☐ Met ☐ |
| **10. Emphasis on Mathematical Reasoning**<br>Materials support the Standards' emphasis on mathematical reasoning, by (all of the following): | Prompting students to construct viable arguments and critique the arguments of others concerning key grade-level mathematics that is detailed in the content standards (cf. MP.3). | | Not Met ☐ Met ☐ |
| | Engaging students in problem solving as a form of argument. | | Not Met ☐ Met ☐ |
| | Explicitly attending to the specialized language of mathematics. | | Not Met ☐ Met ☐ |
| **[S] Consistency with CCSSM** | Materials for science and technical subjects are consistent with CCSSM. | | Not Met ☐ Met ☐ |

# COMMON CORE STATE STANDARDS

## Mathematics

March 2010

# Contents

# Introduction

## Toward greater focus and coherence

*The composite standards [of Hong Kong, Korea and Singapore] have a number of features that can inform an international benchmarking process for the development of K–6 mathematics standards in the US. First, the composite standards concentrate the early learning of mathematics on the number, measurement, and geometry strands with less emphasis on data analysis and little exposure to algebra. The Hong Kong standards for grades 1–3 devote approximately half the targeted time to numbers and almost all the time remaining to geometry and measurement.*

Ginsburg, Leinwand and Decker, 2009

*Mathematics experiences in early childhood settings should concentrate on (1) number (which includes whole number, operations, and relations) and (2) geometry, spatial relations, and measurement, with more mathematics learning time devoted to number than to other topics. The mathematical process goals should be integrated in these content areas. Children should understand the concepts and learn the skills exemplified in the teaching-learning paths described in this report.*

National Research Council, 2009

*In general, the US textbooks do a much worse job than the Singapore textbooks in clarifying the mathematical concepts that students must learn. Because the mathematics concepts in these textbooks are often weak, the presentation becomes more mechanical than is ideal. We looked at both traditional and non-traditional textbooks used in the US and found this conceptual weakness in both.*

Ginsburg et al., 2005

Notable in the research base for these standards are conclusions from TIMSS and other studies of high-performing countries that the traditional US mathematics curriculum must become substantially more coherent and more focused in order to improve student achievement in mathematics. To deliver on the promise of common standards, the standards must address the problem of a curriculum that is 'a mile wide and an inch deep.' The draft Common Core State Standards for Mathematics are a substantial answer to this challenge.

It is important to recognize that "fewer standards" are no substitute for *focused* standards. Achieving "fewer standards" would be easy to do by simply resorting to broad, general statements. Instead, the draft Common Core State Standards for Mathematics aim for clarity and specificity.

Assessing the coherence of a set of standards is more difficult than assessing their focus. William Schmidt and Richard Houang (2002) have said that content standards and curricula are coherent if they are:

> *articulated over time as a sequence of topics and performances that are logical and reflect, where appropriate, the sequential or hierarchical nature of the disciplinary content from which the subject matter derives. That is, what and how students are taught should reflect* not only the topics *that fall within a certain academic discipline,* but also the key ideas *that determine how knowledge is organized and generated within that discipline. This implies that "to be coherent," a set of content standards must evolve from particulars (e.g., the meaning and operations of whole numbers, including simple math facts and routine computational procedures associated with whole numbers and fractions) to deeper structures inherent in the discipline. This deeper structure then serves as a means for connecting the particulars (such as an understanding of the rational number system and its properties).* (emphasis added)

The draft Common Core State Standards for Mathematics endeavor to follow such a design, not only by stressing conceptual understanding of the key ideas, but also by continually returning to organizing principles such as place value or the laws of arithmetic to structure those ideas.

The standards in this draft document define what students should understand and be able to do. Asking a student to understand something means asking a teacher to assess whether the student has understood it. But what does mathematical understanding look like? One hallmark of mathematical understanding is the ability to justify, in a way appropriate to the student's mathematical maturity, *why* a particular mathematical statement is true or where a mathematical rule comes from. There is a world of difference between the student who can summon a mnemonic device such as "FOIL" to expand a product such as $(a + b)(x + y)$ and a student who can explain where that mnemonic comes from. Teachers often observe this difference firsthand, even if large-scale assessments in the year 2010 often do not. The student who can explain the rule understands the mathematics, and may have a better chance to succeed at a less familiar task such as expanding $(a + b + c)(x + y)$. Mathematical understanding and procedural skill are equally important, and both are assessable using mathematical tasks of sufficient richness.

The draft Common Core State Standards for Mathematics begin on the next page with eight Standards for Mathematical Practice. These are not a list of individual math topics, but rather a list of ways in which developing student-practitioners of mathematics increasingly ought to engage with those topics as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years.

*Grateful acknowledgment is here made to Dr. Cathy Kessel for editing the draft standards.*

# Mathematics | Standards for Mathematical Practice

Proficient students of all ages expect mathematics to make sense. They take an active stance in solving mathematical problems. When faced with a non-routine problem, they have the courage to plunge in and try something, and they have the procedural and conceptual tools to continue. They are experimenters and inventors, and can adapt known strategies to new problems. They think strategically.

The practices described below are encouraged in apprentices by expert mathematical thinkers. Students who engage in these practices, individually and with their classmates, discover ideas and gain insights that spur them to pursue mathematics beyond the classroom walls. They learn that effort counts in mathematical achievement. Encouraging these practices in students of all ages should be as much a goal of the mathematics curriculum as the learning of specific content.

## 1 Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

## 2 Reason abstractly and quantitatively.

Mathematically proficient students make sense of the quantities and their relationships in problem situations. Students bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

## 3 Construct viable arguments and critique the reasoning of others.

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

## 4 Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, 2-by-2 tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

## 5 Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, ruler, protractor, calculator, spreadsheet, computer algebra system, statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students interpret graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

## 6 Attend to precision.

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

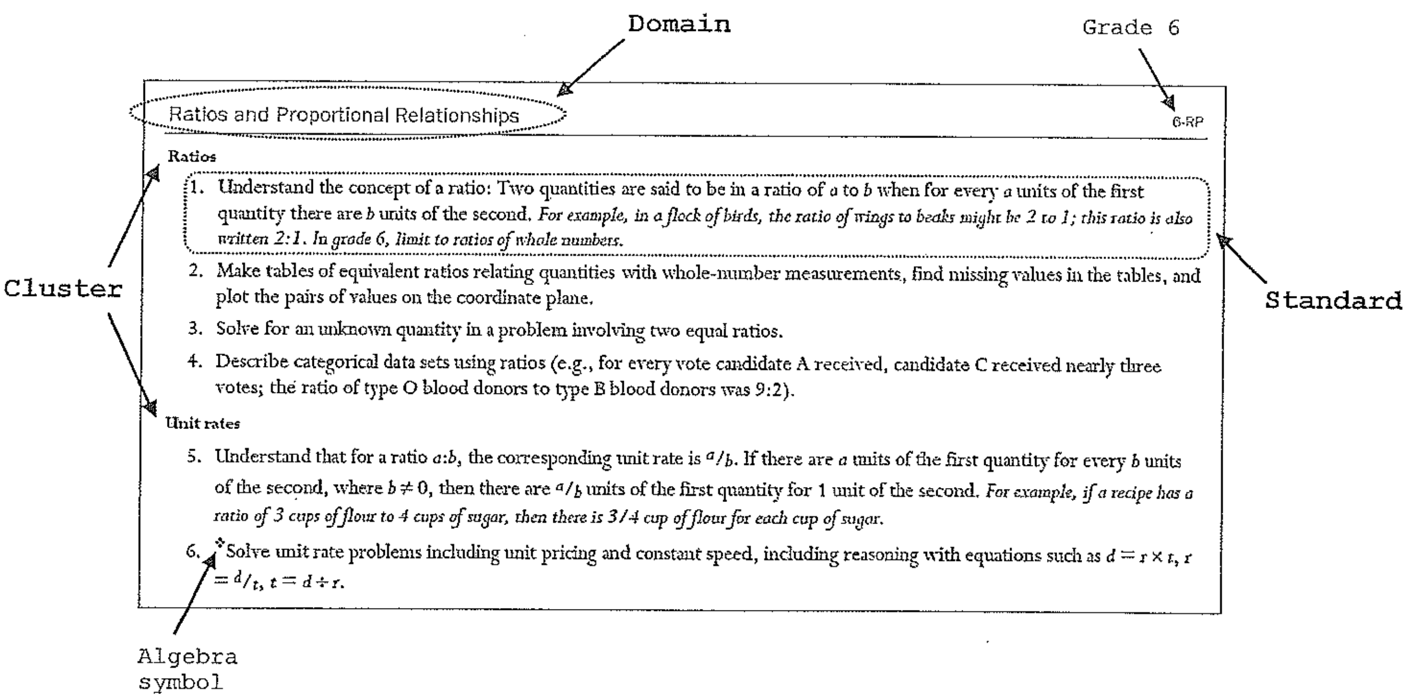
## 7 Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see $7 \times 8$ equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as $2 \times 7$ and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers $x$ and $y$.

## 8 Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

# How to read the grade level standards




Ratios and Proportional Relationships 6-RP

**Ratios**

1. Understand the concept of a ratio: Two quantities are said to be in a ratio of $a$ to $b$ when for every $a$ units of the first quantity there are $b$ units of the second. *For example, in a flock of birds, the ratio of wings to beaks might be 2 to 1; this ratio is also written 2:1. In grade 6, limit to ratios of whole numbers.*
2. Make tables of equivalent ratios relating quantities with whole-number measurements, find missing values in the tables, and plot the pairs of values on the coordinate plane.
3. Solve for an unknown quantity in a problem involving two equal ratios.
4. Describe categorical data sets using ratios (e.g., for every vote candidate A received, candidate C received nearly three votes; the ratio of type O blood donors to type B blood donors was 9:2).

**Unit rates**

5. Understand that for a ratio $a:b$, the corresponding unit rate is $^a/_b$. If there are $a$ units of the first quantity for every $b$ units of the second, where $b \neq 0$, then there are $^a/_b$ units of the first quantity for 1 unit of the second. *For example, if a recipe has a ratio of 3 cups of flour to 4 cups of sugar, then there is 3/4 cup of flour for each cup of sugar.*
6. ❖ Solve unit rate problems including unit pricing and constant speed, including reasoning with equations such as $d = r \times t$, $r = {^d/_t}$, $t = d \div r$.

**Standards** define what students should understand and be able to do. **Clusters** are groups of related standards. Note that standards from different clusters may sometimes be closely related, because mathematics is a connected subject. **Domains** are larger groups of related standards. For each grade level in Grades K–8, the standards are organized into four or five domains. Standards from different domains may sometimes be closely related.

**Algebra Symbol:** Key standards for the development of algebraic thinking in Grades K–5 are indicated by ❖.

**Dotted Underlines:** Dotted underlines, for example, decade words, indicate terms that are explained in the Glossary. In each grade, underlining is used for the first occurrence of a defined term, but not in subsequent occurrences.

**Note on Grade Placement of Topics.** What students can learn at any particular grade level depends upon what they have learned before. Ideally then, each standard in this document might have been phrased in the form, "Students who already know A should next come to learn B." But in the year 2010 this approach is unrealistic—not least because existing education research cannot specify all such learning pathways. Of necessity therefore, grade placements for specific topics have been made on the basis of state and international comparisons and the collective experience and collective professional judgment of educators, researchers and mathematicians. One promise of common state standards is that over time they will allow research on learning progressions to inform and improve the design of standards to a much greater extent than is possible today. Learning opportunities will continue to vary across schools and school systems, and educators should make every effort to meet the needs of individual students based on their current understanding.

**Note on Ordering of Topics within a Grade.** These standards do not dictate curriculum. In particular, just because topic A appears before topic B in the standards for a given grade, it does not necessarily mean that topic A must be taught before topic B. A teacher might prefer to teach topic B before topic A, or might choose to highlight connections by teaching topic A and topic B at the same time. Or, a teacher might prefer to teach a topic of his or her own choosing that leads, as a byproduct, to students reaching the standards for topics A and B.

# Overview of the Mathematics Standards
## Grades K–5

This table shows the domains and clusters in each grade K–5

| | *K* | *1* | *2* | *3* | *4* | *5* |
|---|---|---|---|---|---|---|
| Number—Counting and Cardinality | • Number names<br>• Counting to tell the number of objects<br>• Comparing and ordering numbers | | | | | |
| Number—Operations and the Problems They Solve | • Composing and decomposing numbers; addition and subtraction | • Addition and subtraction<br>• Describing situations and solving problems with addition and subtraction | • Addition and subtraction<br>• Describing situations and solving problems with addition and subtraction | • Multiplication and division<br>• Describing situations and solving problems with multiplication and division | • Multiplication and Division<br>• Problem solving with the four operations | |
| Number—Base Ten | • Two-digit numbers<br>• Composing and decomposing ten | • Numbers up to 100<br>• Adding and subtracting in base ten | • Numbers up to 1000<br>• Adding and subtracting in base ten | • Numbers up to 10,000<br>• Adding and subtracting in base ten<br>• Multiplying and dividing in base ten | • Numbers up to 100,000<br>• Multiplying and dividing in base ten | • Whole numbers in base ten<br>• Decimal concepts<br>• Operations on decimals |
| Number—Fractions | | | | • Fractions as representations of numbers<br>• Fractional quantities | • Operations on fractions<br>• Decimal concepts | • Fraction equivalence<br>• Operations on fractions |
| Measurement and Data | • Direct measurement<br>• Representing and interpreting data | • Length measurement<br>• Time measurement<br>• Representing and interpreting data | • Length measurement<br>• Time and money<br>• Representing and interpreting data | • The number line and units of measure<br>• Perimeter and area<br>• Representing and interpreting data | • The number line and units of measure<br>• Perimeter and area<br>• Angle measurement<br>• Representing and interpreting data | • Units of measure<br>• Volume<br>• Representing and interpreting data |
| Geometry | • Shapes, their attributes, and spatial reasoning | • Shapes, their attributes, and spatial reasoning | • Shapes, their attributes, and spatial reasoning | • Properties of 2-dimensional shapes<br>• Structuring rectangular shapes | • Lines and angles<br>• Line symmetry | • Coordinates<br>• Plane figures |

# Overview of the Mathematics Standards
## Grades 6–8

This table shows the domains and clusters in each grade 6–8.

| | Grade 6 | Grade 7 | Grade 8 |
|---|---|---|---|
| Ratios and Proportional Relationships | • Ratios<br>• Unit rates | • Analyzing proportional relationships<br>• Percent | |
| The Number System | • Operations<br>• The system of rational numbers | • The system of rational numbers<br>• The system of real numbers | • The system of real numbers |
| Expressions and Equations | • Expressions<br>• Quantitative relationships and the algebraic approach to problems | • Expressions<br>• Quantitative relationships and the algebraic approach to solving problems | • Slopes of lines in the coordinate plane<br>• Linear equations and systems |
| Functions | | | • Function concepts<br>• Functional relationships between quantities |
| Geometry | • Properties of area, surface area, and volume | • Congruence and similarity<br>• Angles | • Congruence and similarity<br>• The Pythagorean Theorem<br>• Plane and solid geometry |
| Statistics and Probability | • Variability and measures of center<br>• Summarizing and describing distributions | • Situations involving randomness<br>• Random sampling to draw inferences about a population<br>• Comparative inferences about two populations | • Patterns of association in bivariate data |

# Mathematics | Kindergarten

In Kindergarten, instructional time should focus on two critical areas: (1) representing, comparing and ordering whole numbers and joining and separating sets; (2) describing shapes and space. More learning time in Kindergarten should be devoted to number than to other topics.

(1) Students use numbers, including written numerals, to represent quantities and to solve quantitative problems, such as counting objects in a set; creating a set with a given number of objects; comparing and ordering sets or numerals; and modeling simple joining and separating situations with objects. They choose, combine, and apply effective strategies for answering quantitative questions, including quickly recognizing the cardinalities of small sets of objects, counting and producing sets of given sizes, counting the number of objects in combined sets, or counting the number of objects that remain in a set after some are taken away.

(2) Students describe their physical world using geometric ideas (e.g., shape, orientation, spatial relations) and vocabulary. They identify, name, and describe basic shapes, such as squares, triangles, circles, rectangles, (regular) hexagons, and (isosceles) trapezoids, presented in a variety of ways (e.g., with different sizes or orientations), as well as three-dimensional shapes such as spheres, cubes, and cylinders. They use basic shapes and spatial reasoning to model objects in their environment and to construct more complex shapes.

**Number names**

1. Say the number name sequence to 100.
2. Know the decade words to ninety and recite them in order ("ten, twenty, thirty, …").
3. Say the number name sequence forward or backward beginning from a given number within the known sequence (instead of always beginning at 1).
4. Write numbers from 1 to 20 in base-ten notation.

**Counting to tell the number of objects**

5. Count to answer "how many?" questions about as many as 20 things. *Objects may be arranged in a line, a rectangular array, a circle, or a scattered configuration.*
6. Understand that when counting objects,
   a. The number names are said in the standard order.
   b. Each object is paired with one and only one number name.
   c. The last number name said tells the number of objects counted.
7. Understand that when counting forward, each successive number name refers to a quantity that is 1 larger.

**Comparing and ordering numbers**

8. Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies. *Include groups with up to ten objects.*
9. Compare and put in order numbers between 1 and 10 presented in written symbols: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

## Number—Operations and the Problems They Solve K-NOP

**Composing and decomposing numbers; addition and subtraction**

1. Understand addition as putting together—e.g., finding the number of objects in a group formed by putting two groups together. Understand subtraction as taking apart—e.g., finding the number of objects left when a one group is taken from another.
2. Represent addition and subtraction with objects, fingers, mental images, drawings, sounds (e.g., claps), acting out situations, verbal explanations, expressions, or equations. *Note that drawings need not show details, but should show the mathematics in the problem. (This note also applies wherever drawings are mentioned in subsequent standards.)*
3. ❖Decompose numbers less than or equal to 10 into pairs in various ways, e.g., using objects or drawings, and record each decomposition by a drawing or equation (e.g., $5 = 2 + 3$). Compose numbers whose sum is less than or equal to 10, e.g., using objects or drawings, and record each composition by a drawing or equation (e.g., $3 + 1 = 4$).❖
4. Compose and decompose numbers less than or equal to 10 in two different ways, and record compositions and decompositions by drawings or equations. *For example, 7 might be composed or decomposed in two different ways by a drawing showing how a group of 2 and a group of 5 together make the same number as do a group of 3 and a group of 4.*
5. ❖Understand that addition and subtraction are related. *For example, when a group of 9 is decomposed into a group of 6 and a group of 3, this means not only $9 = 6 + 3$ but also $9 - 3 = 6$ and $9 - 6 = 3$.*
6. ❖Solve addition and subtraction word problems, and calculate additions and subtractions within 10, e.g., using objects or drawings to represent the problem.
7. Fluently add and subtract, for sums and minuends of 5 or less.

## Number—Base Ten K-NBT

**Two-digit numbers**

1. Understand that 10 can be thought of as a bundle of ones—a unit called a "ten."
2. Understand that a teen number is composed of a ten and one, two, three, four, five, six, seven, eight, or nine ones.
3. Compose and decompose teen numbers into a ten and some ones, e.g., by using objects or drawings, and record the compositions and decompositions in base-ten notation. *For example, $10 + 8 = 18$ and $14 = 10 + 4$.*
4. Put in order numbers presented in base-ten notation from 1 to 20 (inclusive), and be able to explain the reasoning.
5. Understand that a decade word refers to one, two, three, four, five, six, seven, eight, or nine tens.
6. Understand that the two digits of a two-digit number represent amounts of tens and ones. *In 29, for example, the 2 represents two tens and the 9 represents nine ones.*

**Composing and decomposing ten**

7. Decompose 10 into pairs of numbers, e.g., by using objects or drawings, and record each decomposition with a drawing or equation.
8. Compose numbers to make 10, e.g., by using objects or drawings, and record each composition with a drawing or equation.
9. ❖For any number from 1 to 9, find the number that makes 10 when added to the given number, e.g., by using objects or drawings, and record the answer with a drawing or equation.

## Measurement and Data — K-MD

**Direct measurement**

1. Understand that objects have measurable attributes, such as length or weight. A single object might have several measurable attributes of interest.
2. Directly compare two objects with a measurable attribute in common, to see which object has "more of" the attribute. *For example, directly compare the heights of two books and identify which book is taller.*

**Representing and interpreting data**

3. Classify objects or people into given categories; count the numbers in each category and sort the categories by count. *Limit category counts to be less than or equal to 10.*

## Geometry — K-G

**Shapes, their attributes, and spatial reasoning**

1. Describe objects in the environment using names of shapes, and describe the relative positions of these objects using terms such as *above*, *below*, *beside*, *in front of*, *behind*, and *next to*.
2. Understand that names of shapes apply regardless of the orientation or overall size of the shape. *For example, a square in any orientation is still a square. Students may initially need to physically rotate a shape until it is "level" before they can correctly name it.*
3. Understand that shapes can be two-dimensional (lying in a plane, "flat") or three-dimensional ("solid").
4. Understand that shapes can be seen as having parts, such as sides and vertices ("corners"), and that shapes can be put together to compose other shapes.
5. Analyze and compare a variety of two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, component parts (e.g., number of sides and vertices) and other attributes (e.g., having sides of equal length).
6. Combine two- or three-dimensional shapes to solve problems such as deciding which puzzle piece will fit into a place in a puzzle.

# Mathematics | Grade 1

In Grade 1, instructional time should focus on four critical areas: (1) developing understanding of addition, subtraction, and strategies for additions and subtractions within 20; (2) developing understanding of whole number relationships, including grouping in tens and ones, (3) developing understanding of linear measurement and measuring lengths, and (4) composing and decomposing geometric shapes.

(1) Students develop strategies for adding and subtracting whole numbers based on their prior work with small numbers. They use a variety of models, including discrete objects and length-based models (e.g., cubes connected to form lengths), to model "put together/take apart," "add to," "take from," and "compare" situations to develop meaning for the operations of addition and subtraction, and to develop strategies to solve arithmetic problems with these operations. Students understand connections between counting and addition and subtraction (i.e., adding two is the same as counting on two). They use properties of addition (commutativity and associativity) to add whole numbers and to create and use increasingly sophisticated strategies based on these properties (e.g., "making tens") to solve addition and subtraction problems within 20. By comparing a variety of solution strategies, children build their understanding of the inverse relationship between addition and subtraction.

(2) Students compare and order whole numbers (at least to 100), to develop understanding of and solve problems involving their relative sizes. They think of whole numbers between 10 and 100 in terms of tens and ones (especially recognizing the numbers 11 to 19 as composed of a ten and some ones). They understand the sequential order of the counting numbers and their relative magnitudes through activities such as representing numbers on paths of numbered things.

(3) Students develop an understanding of the meaning and processes of measurement, including underlying concepts such as partitioning (the mental activity of decomposing the length of an object into equal-sized units) and transitivity (e.g., in terms of length, if object A is longer than object B and object B is longer than object C, then object A is longer than object C). They understand linear measure as an iteration of units, and use rulers and other measurement tools with that understanding.

(4) Students compose and decompose plane and solid figures (e.g., put two congruent isosceles triangles together to make a rhombus), building understanding of part-whole relationships as well as the properties of the original and composite shapes. As they combine solid and plane figures, they recognize them from different perspectives and orientations, describe their geometric attributes, and determine how they are alike and different, to develop the background for measurement and for initial understandings of properties such as congruence and symmetry.

**Addition and subtraction**

1. ❖Understand the properties of addition.
   a. Addition is commutative. *For example, if 3 cups are added to a stack of 8 cups, then the total number of cups is the same as when 8 cups are added to a stack of 3 cups; that is, $8 + 3 = 3 + 8$.*
   b. Addition is associative. *For example, $4 + 3 + 2$ can be found by first adding $4 + 3 = 7$ then adding $7 + 2 = 9$, or by first adding $3 + 2 = 5$ then adding $4 + 5 = 9$.*
   c. 0 is the additive identity.
2. ❖Explain and justify properties of addition and subtraction, e.g., by using representations such as objects, drawings, and story contexts. Explain what happens when:
   a. The order of addends in a sum is changed in a sum with two addends.
   b. 0 is added to a number.
   c. A number is subtracted from itself.
   d. One addend in a sum is increased by 1 and the other addend is decreased by 1. *Limit to two addends.*
3. ❖Understand that addition and subtraction have an inverse relationship. *For example, if $8 + 2 = 10$ is known, then $10 - 2 = 8$ and $10 - 8 = 2$ are also known.*
4. ❖Understand that when all but one of three numbers in an addition or subtraction equation are known, the unknown number can be found. *Limit to cases where the unknown number is a whole number.*
5. Understand that addition can be recorded by an expression (e.g., $6 + 3$), or by an equation that shows the sum (e.g., $6 + 3 = 9$). Likewise, subtraction can be recorded by an expression (e.g., $9 - 5$), or by an equation that shows the difference (e.g., $9 - 5 = 4$).

**Describing situations and solving problems with addition and subtraction**

6. Understand that addition and subtraction apply to situations of adding-to, taking-from, putting together, taking apart, and comparing. *See Glossary, Table 1.*
7. ❖Solve word problems involving addition and subtraction within 20, e.g., by using objects, drawings and equations to represent the problem. *Students should work with all of the addition and subtraction situations shown in the Glossary, Table 1, solving problems with unknowns in all positions, and representing these situations with equations that use a symbol for the unknown (e.g., a question mark or a small square). Grade 1 students need not master the more difficult problem types.*
8. Solve word problems involving addition of three whole numbers whose sum is less than or equal to 20.

## Number—Base Ten 1-NBT

**Numbers up to 100**

1. Read and write numbers to 100.
2. Starting at any number, count to 100 or beyond.
3. Understand that when comparing two-digit numbers, if one number has more tens, it is greater; if the amount of tens is the same in each number, then the number with more ones is greater.
4. Compare and order two-digit numbers based on meanings of the tens and ones digits, using > and < symbols to record the results of comparisons.

**Adding and subtracting in base ten**

5. Calculate mentally, additions and subtractions within 20.
   a. Use strategies that include counting on; making ten (for example, $7 + 6 = 7 + 3 + 3 = 10 + 3 = 13$); and decomposing a number (for example, $17 - 9 = 17 - 7 - 2 = 10 - 2 = 8$).
6. Demonstrate fluency in addition and subtraction within 10.
7. Understand that in adding or subtracting two-digit numbers, one adds or subtracts like units (tens and tens, ones and ones) and sometimes it is necessary to compose or decompose a higher value unit.
8. Given a two-digit number, mentally find 10 more or 10 less than the number, without having to count.
9. Add one-digit numbers to two-digit numbers, and add multiples of 10 to one-digit and two-digit numbers.
10. Explain addition of two-digit numbers using concrete models or drawings to show composition of a ten or a hundred.
11. ❖Add two-digit numbers to two-digit numbers using strategies based on place value, properties of operations, and/or the inverse relationship between addition and subtraction; explain the reasoning used.

## Measurement and Data

1-MD

**Length measurement**

1. Order three objects by length; compare the length of two objects indirectly by using a third object.
2. Understand that the length of an object can be expressed numerically by using another object as a length unit (such as a paper-clip, yardstick, or inch length on a ruler). The object to be measured is partitioned into as many equal parts as possible with the same length as the length unit. The length measurement of the object is the number of length units that span it with no gaps or overlaps. *For example, "I can put four paperclips end to end along the pencil, so the pencil is four paperclips long."*
3. Measure the length of an object by using another object as a length unit.

**Time measurement**

4. Tell time from analog clocks in hours and half- or quarter-hours.

**Representing and interpreting data**

5. Organize, represent, and interpret data with several categories; ask and answer questions about the total number of data points, how many in each category, and how many more or less are in one category than in another.

## Geometry

1-G

**Shapes, their attributes, and spatial reasoning**

1. Distinguish between defining attributes (e.g., triangles are closed and three-sided) versus non-defining attributes (e.g., color, orientation, overall size) for a wide variety of shapes.
2. Understand that shapes can be joined together (composed) to form a larger shape or taken apart (decomposed) into a collection of smaller shapes. Composing multiple copies of some shapes creates tilings. *In this grade, "circles," "rectangles," and other shapes include their interiors as well as their boundaries.*
3. Compose two-dimensional shapes to create a unit, using cutouts of rectangles, squares, triangles, half-circles, and quarter-circles. Form new shapes by repeating the unit.
4. Compose three-dimensional shapes to create a unit, using concrete models of cubes, right rectangular prisms, right circular cones, and right circular cylinders. Form new shapes by repeating the unit. *Students do not need to learn formal names such as "right rectangular prism."*
5. Decompose circles and rectangles into two and four equal parts. Describe the parts using the words *halves*, *fourths*, and *quarters*, and using the phrases *half of*, *fourth of*, and *quarter of*. Describe the whole as two of, or four of the parts. Understand that decomposing into more equal shares creates smaller shares.
6. Decompose two-dimensional shapes into rectangles, squares, triangles, half-circles, and quarter-circles, including decompositions into equal shares.

# Mathematics | Grade 2

In Grade 2, instructional time should focus on three critical areas: (1) developing understanding of base-ten notation; (2) developing fluency with additions and subtractions within 20 and fluency with multi-digit addition and subtraction; and (3) describing and analyzing shapes.

(1) Students develop an understanding of the base-ten system (at least to 1000). Their understanding of the base-ten system includes ideas of counting in units (twos, fives, and tens) and multiples of hundreds, tens, and ones, as well as number relationships, including comparing and ordering. They understand multi-digit numbers (up to 1000) written in base-ten notation, recognizing that the digits in each place represent thousands, hundreds, tens, or ones (e.g., 853 is 8 hundreds + 5 tens + 3 ones).

(2) Students use their understanding of addition to develop fluency with additions and subtractions within 20. They solve arithmetic problems by applying their understanding of models for addition and subtraction (such as combining or separating sets or using number lines that begin with zero), relationships and properties of numbers, and properties of addition. They develop, discuss, and use efficient, accurate, and generalizable methods to compute sums and differences of two-digit whole numbers. They select and accurately apply methods that are appropriate for the context and the numbers involved to mentally calculate sums and differences. They develop fluency with efficient procedures, including standard algorithms, for adding and subtracting whole numbers; understand and explain why the procedures work based on their understanding of base-ten notation and properties of operations; and use them to solve problems.

(3) Students describe and analyze shapes by examining their sides and angles. Students investigate, describe, and reason about decomposing and combining shapes to make other shapes. Through building, drawing, and analyzing two- and three-dimensional shapes, students develop a foundation for understanding attributes of two- and three-dimensional space such as area and volume, and properties such as congruence and symmetry that they will learn about in later grades.

**Addition and subtraction**

1. ✣Explain and justify properties of addition and subtraction, e.g., by using representations such as objects, drawings, and story contexts. Include properties such as:
   a. Changing the order of addends does not change their sum.
   b. Subtracting one addend from a sum of two numbers results in the other addend.
   c. If more is subtracted from a number, the difference is decreased, and if less is subtracted the difference is increased.
   d. In an addition equation, each addend can be decomposed and the parts can be recombined in any order without changing the sum. *For example, $5 + 3 = 8$. Because 5 decomposes as $4 + 1$, the first addend can be replaced by $4 + 1$, yielding $(4 + 1) + 3 = 8$. Recombining in two different orders: $4 + 4 = 8$, also $7 + 1 = 8$.*

**Describing situations and solving problems with addition and subtraction**

2. ✣Solve word problems involving addition and subtraction within 100, e.g., by using drawings or equations to represent the problem. *Students should work with all of the addition and subtraction situations shown in the Glossary, Table 1, solving problems with unknown sums, addends, differences, minuends, and subtrahends, and representing these situations with equations that use a symbol for the unknown (e.g., a question mark or a small square). Focus on the more difficult problem types.*
3. Solve two-step word problems involving addition and subtraction within 100, e.g., by using drawings or equations to represent the problem.

## Number—Base Ten 2-NBT

**Numbers up to 1000**

1. Understand that 100 can be thought of as a bundle of tens—a unit called a "hundred."
2. Read and write numbers to 1000 using base-ten notation, number names, and expanded form.
3. Count within 1000; skip count by 2s, 5s, 10s, and 100s.
4. Understand that when comparing three-digit numbers, if one number has more hundreds, it is greater; if the amount of hundreds is the same in each number, then the number with more tens is greater. If the amount of tens and hundreds is the same in each number, then the number with more ones is greater.
5. Compare and order three-digit numbers based on meanings of the hundreds, tens, and ones digits.

**Adding and subtracting in base ten**

6. Fluently add and subtract within 20. By end of Grade 2, know from memory sums of one-digit numbers.
7. Mentally compute sums and differences of multiples of 10. *For example, mentally calculate $130 - 80$.*
8. Understand that in adding or subtracting three-digit numbers, one adds or subtracts like units (hundreds and hundreds, tens and tens, ones and ones) and sometimes it is necessary to compose or decompose a higher value unit.
9. Given a number from 100 to 900, mentally find 10 more or 10 less than the number, and mentally find 100 more or 100 less than the number, without counting.
10. Understand that algorithms are predefined steps that give the correct result in every case, while strategies are purposeful manipulations that may be chosen for specific problems, may not have a fixed order, and may be aimed at converting one problem into another. *For example, one might mentally compute $503 - 398$ as follows: $398 + 2 = 400$, $400 + 100 = 500$, $500 + 3 = 503$, so the answer is $2 + 100 + 3$, or 105.*
11. ✣ Compute sums and differences of one-, two-, and three-digit numbers using strategies based on place value, properties of operations, and/or the inverse relationship between addition and subtraction; explain the reasoning used.
12. ✣Explain why addition and subtraction strategies and algorithms work, using place value and the properties of operations. *Include explanations supported by drawings or objects. A range of reasonably efficient algorithms may be covered, not only the standard algorithm.*
13. Compute sums of two three-digit numbers, and compute sums of three or four two-digit numbers, using the standard algorithm; compute differences of two three-digit numbers using the standard algorithm.

## Measurement and Data 2-MD

**Length measurement**

1. Understand that 1 inch, 1 foot, 1 centimeter, and 1 meter are conventionally defined lengths used as standard units.
2. Measure lengths using measurement tools such as rulers, yardsticks and measuring tapes; understand that these tools are used to find out how many standard length units span an object with no gaps or overlaps, when the 0 mark of the tool is aligned with an end of the object.

3. Understand that when measuring a length, if a smaller unit is used, more copies of that unit are needed to measure the length than would be necessary if a larger unit were used.
4. Understand that units can be decomposed into smaller units, e.g., 1 foot can be decomposed into 12 inches and 1 meter can be decomposed into 100 centimeters. A small number of long units might compose a greater length than a large number of small units.
5. Understand that lengths can be compared by placing objects side by side, with one end lined up. The difference in lengths is how far the longer extends beyond the end of the shorter.
6. Understand that a sum of two whole numbers can represent a combination of two lengths; a difference of two whole numbers can represent a difference in length; find total lengths and differences in lengths using addition and subtraction.

**Time and money**

7. Find time intervals between hours in one day.
8. Solve word problems involving dollar bills, quarters, dimes, nickels and pennies. *Do not include dollars and cents in the same problem.*

**Representing and interpreting data**

9. Generate measurement data by measuring whole-unit lengths of several objects, or by making repeated measurements of the same object. Show the measurements by making a dot plot, where the horizontal scale is marked off in whole-number units.
10. Draw a picture graph and a bar graph (with single-unit scale) to represent a data set with several categories. Connect representations on bar graph scales, rulers, and number lines that begin with zero. Solve simple Put Together/Take Apart and Compare problems using information presented in a bar graph. *See Glossary, Table 1.*

## Geometry 2-G

**Shapes, their attributes, and spatial reasoning**

1. Understand that different categories of shapes (e.g., rhombuses, trapezoids, rectangles, and others) can be united into a larger category (e.g., quadrilaterals) on the basis of shared attributes (e.g., having four straight sides).
2. Identify and name polygons of up to six sides by the number of their sides or angles.
3. Recognize rectangles, rhombuses, squares and trapezoids as examples of quadrilaterals; draw examples of quadrilaterals that do not belong to any of these subcategories.
4. Draw and identify shapes that have specific attributes, such as number of equal sides or number of equal angles. *Sizes of lengths and angles are compared directly or visually, not compared by measuring.*
5. Recognize objects as resembling spheres, right circular cylinders, and right rectangular prisms. *Students do not need to learn formal names such as "right rectangular prism."*
6. Decompose circular and rectangular objects into two, three, or four equal parts. Describe the parts using the words *halves, thirds, half of, a third of,* etc.; describe the wholes as two halves, three thirds, four fourths. Recognize that a half, a third, or a fourth of a circular or rectangular object—a graham cracker, for example—is the same size regardless of its shape.

# Mathematics | Grade 3

In Grade 3, instructional time should focus on four critical areas: (1) developing understanding of multiplication and division and strategies for multiplication and division within 100; (2) developing understanding of fractions, starting with unit fractions; (3) developing understanding of the structure of rectangular arrays and of area; and (4) describing and analyzing two-dimensional shapes. Multiplication, division, and fractions are the most important developments in Grade 3.

(1) Students develop an understanding of the meanings of multiplication and division of whole numbers through the use of representations such as equal-sized groups, arrays, area models, and equal jumps on number lines for multiplication; and successive subtraction, partitioning, and sharing for division. Through this process, numbers themselves take on new meaning and are no longer only counters for single objects. They represent groups, a number of groups (for example, 3 teams of 6 people), or a comparative factor (3 times as long).

Students use properties of operations to calculate products of whole numbers. They use increasingly sophisticated strategies based on these properties to solve multiplication and division problems involving single-digit factors. By comparing a variety of solution strategies, students learn the inverse relationship between multiplication and division.

(2) Students develop an understanding of a definition of a fraction, beginning with unit fractions. They use fractions to represent parts of a whole or distances on a number line that begins with zero. Students understand that the size of a fractional part is relative to the size of the whole (for example, ¼ of a mile is longer than ¾ of a foot, even though $\frac{1}{4} < \frac{3}{4}$), and they are able to use fractions to represent numbers equal to, less than, and greater than one. They solve problems that involve comparing and ordering fractions using by models or strategies based on noticing common numerators or denominators.

(3) Students recognize area as an attribute of two-dimensional regions. They understand that area can be quantified by finding the total number of same-size units of area required to cover the shape without gaps or overlaps. They understand that a 1-unit by 1-unit square is the standard unit for measuring area. Students understand that rectangular arrays can be decomposed into identical rows or into identical columns. By decomposing rectangles into rectangular arrays of squares, students connect area measure to the area model used to represent multiplication, and they use this connection to justify using multiplication to determine the area of a rectangle. Students contrast area with perimeter.

(4) Students describe, analyze, and compare properties of two-dimensional shapes. They compare and classify the shapes by their sides and angles, and connect these with definitions of shapes. Students investigate, describe, and reason about decomposing and combining polygons to make other polygons. Through building, drawing, and analyzing two-dimensional shapes, students deepen their understanding of attributes and properties of two-dimensional objects.

**Multiplication and division**

1. Understand that multiplication of whole numbers is repeated addition. *For example, 5 × 7 means 7 added to itself 5 times. Products can be represented by rectangular arrays, with one factor the number of rows and the other the number of columns.*
2. ❖Understand the properties of multiplication.
   a. Multiplication is commutative. *For example, the total number in 3 groups with 6 things each is the same as the total number in 6 groups with 3 things each, that is,* $3 \times 6 = 6 \times 3$.
   b. Multiplication is associative. *For example,* $4 \times 3 \times 2$ *can be calculated by first calculating* $4 \times 3 = 12$ *then calculating* $12 \times 2 = 24$, *or by first calculating* $3 \times 2 = 6$ *then calculating* $4 \times 6 = 24$.
   c. 1 is the multiplicative identity.
   d. Multiplication distributes over addition (the distributive property). *For example,* $5 \times (3 + 4) = (5 \times 3) + (5 \times 4)$.
3. ❖Explain and justify properties of multiplication and division, e.g., by using representations such as objects, drawings, and story contexts. Include properties such as:
   a. Changing the order of two factors does not change their product.
   b. The product of a number and 1 is the number.
   c. Dividing a nonzero number by itself yields 1.
   d. Multiplying a quantity by a nonzero number, then dividing by the same number, yields the original quantity.
   e. When one factor in a product is multiplied by a number and another factor divided by the same number, the product is unchanged. *Limit to multiplying and dividing by numbers that result in whole-number quotients.*
   f. Products where one factor is a one-digit number can be computed by decomposing one factor as the sum of two numbers, multiplying each number by the other factor, and adding the two products.
4. ❖Understand that multiplication and division have an inverse relationship. *For example, if* $5 \times 7 = 35$ *is known, then* $35 \div 5 = 7$ *and* $35 \div 7 = 5$ *are also known. The division* $35 \div 5$ *means the number which yields 35 when multiplied by 5; because* $5 \times 7 = 35$, *then* $35 \div 5 = 7$.
5. ❖Understand that when all but one of three numbers in a multiplication or division equation are known, the unknown number can be found. *Limit to cases where the unknown number is a whole number.*

**Describing situations and solving problems with multiplication and division**

6. Understand that multiplication and division apply to situations with equal groups, arrays or area, and comparing. *See Glossary, Table 2.*
7. ❖Solve word problems involving multiplication and division within 100, using an equation with a symbol for the unknown to represent the problem. *This standard is limited to problems with whole-number quantities and whole-number quotients. Focus on situations described in the Glossary, Table 2.*
8. ❖Solve one- or two-step word problems involving the four operations. *This standard is limited to problems with whole-number quantities and whole-number quotients.*
9. Understand that multiplication and division can be used to compare quantities (see Glossary, Table 2); solve multiplicative comparison problems with whole numbers (problems involving the notion of "times as much").

## Number—Base Ten 3-NBT

**Numbers up to 10,000**

1. Understand that 1000 can be thought of as a bundle of hundreds—a unit called a "thousand."
2. Read and write numbers to 10,000 using base-ten notation, number names, and expanded form.
3. Count within 10,000; skip count by 10s, 100s and 1000s.
4. Understand that when comparing four-digit numbers, if one number has more thousands, it is greater; if the amount of thousands is the same in each number, then the number with more hundreds is greater; and so on. Compare and order four-digit numbers based on meanings of the digits.

**Adding and subtracting in base ten**

5. Mentally calculate sums and differences of multiples of 10, 100, and 1000. *For example, mentally calculate* $1300 - 800$
6. Given a number from 1000 to 9000, mentally find 100 more or 100 less than the number, and mentally find 1000 more or 1000 less than the number, without counting.

**Multiplying and dividing in base ten**

7. ❖Understand that the distributive property is at the heart of strategies and algorithms for multiplication and division computations with numbers in base-ten notation; use the distributive property and other properties of operations to explain patterns in the multiplication table and to derive new multiplication and division equations from known ones. *For example, the distributive property makes it possible to multiply* $4 \times 7$ *by decomposing 7 as* $5 + 2$ *and using* $4 \times 7 = 4 \times (5 + 2) = (4 \times 5) + (4 \times 2) = 20 + 8 = 28$.
8. Fluently multiply one-digit numbers by 10.
9. Use a variety of strategies for multiplication and division within 100. By end of Grade 3, know from memory products of one-digit numbers where one of the factors is 2, 3, 4, or 5.

## Number—Fractions 3-NF

**Fractions as representations of numbers**

1. Understand that a unit fraction corresponds to a point on a number line. *For example, 1/3 represents the point obtained by decomposing the interval from 0 to 1 into three equal parts and taking the right-hand endpoint of the first part. In Grade 3, all number lines begin with zero.*
2. Understand that fractions are built from unit fractions. *For example, 5/4 represents the point on a number line obtained by marking off five lengths of ¼ to the right of 0.*
3. Understand that two fractions are equivalent (represent the same number) when both fractions correspond to the same point on a number line. Recognize and generate equivalent fractions with denominators 2, 3, 4, and 6 (e.g., $1/2 = 2/4$, $4/6 = 2/3$), and explain the reasoning.
4. Understand that whole numbers can be expressed as fractions. *Three important cases are illustrated by the examples* $1 = 4/4$, $6 = 6/1$, *and* $7 = (4 \times 7)/4$. *Expressing whole numbers as fractions can be useful for solving problems or making calculations.*

**Fractional quantities**

5. Understand that fractions apply to situations where a whole is decomposed into equal parts; use fractions to describe parts of wholes. *For example, to show 1/3 of a length, decompose the length into 3 equal parts and show one of the parts.*
6. Compare and order fractional quantities with equal numerators or equal denominators, using the fractions themselves, tape diagrams, number line representations, and area models. Use > and < symbols to record the results of comparisons.

## Measurement and Data 3-MD

**The number line and units of measure**

1. Understand that a number line has an origin (0) and a unit (1), with whole numbers one unit distance apart. Use number lines to represent problems involving distances, elapsed time, amounts of money and other quantities. *In such problems, the interval from 0 to 1 may represent a unit of distance, time, money, etc.*
2. Understand that a unit of measure can be decomposed into equal-sized parts, whose sizes can be represented as fractions of the unit. Convert measurements in one unit to measurements in a smaller or a larger unit, and solve problems involving such mixed units (e.g., feet and inches, weeks and days).

**Perimeter and area**

3. Understand and use concepts of area measurement.
   a. A square with side length 1 unit, called "a unit square," is said to have "one square unit" of area, and can be used to measure area.
   b. A plane figure which can be covered without gaps or overlaps by $n$ unit squares has an area of $n$ square units. Areas of some other figures can be measured by using fractions of unit squares or using figures whose areas have been found by decomposing other figures.
   c. When measuring an area, if a smaller unit of measurement is used, more units must be iterated to measure the area in those units.
   d. Determine and compare areas by counting square units. *Use* $cm^2$, $m^2$, $in^2$, $ft^2$, *and improvised units.*
4. Understand that multiplication of whole numbers can be represented by area models; a rectangular region that is $a$ length units by $b$ length units (where $a$ and $b$ are whole numbers) and tiled with unit squares illustrates why the rectangle encloses an area of $a \times b$ square units.
5. Solve problems involving perimeters of polygons.
   a. Add given side lengths, and multiply for the case of equal side lengths.
   b. ❖Find an unknown length of a side in a polygon given the perimeter and all other side lengths; represent these problems with equations involving a letter for the unknown quantity.
   c. Exhibit rectangles with the same perimeter and different area, and with the same area and different perimeter.

**Representing and interpreting data**

6. Draw a scaled picture graph and a scaled bar graph to represent a data set with several categories. Solve one- and two-step "how many more" and "how many less" problems using information presented in scaled bar graphs. *Include single-unit scales and multiple-unit scales; for example, each square in the bar graph might represent 1 pet, 5 pets, or 10 pets.*
7. Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch. Show the data by making a dot plot, where the horizontal scale is marked off in appropriate units—whole numbers, halves, or quarters.

## Geometry 3-G

**Properties of 2-dimensional shapes**

1. Understand that a given category of plane figures (e.g., triangles) has subcategories (e.g., isosceles triangles) defined by special properties.
2. Describe, analyze, compare and classify two-dimensional shapes by their properties and connect these properties to the classification of shapes into categories and subcategories (e.g., squares are "special rectangles" as well as "special rhombuses"). *Focus on triangles and quadrilaterals.*

**Structuring rectangular shapes**

3. Understand that rectangular regions can be tiled with squares in rows and columns, or decomposed into such arrays.
4. Structure a rectangular region spatially by decomposing it into rows and columns of squares. Determine the number of squares in the region using that spatial structure (e.g., by multiplication or skip counting).
5. Understand that shapes can be decomposed into parts with equal areas; the area of each part is a unit fraction of the whole. *For example, when a shape is partitioned into 4 parts with equal area, the area of each part is ¼ of the area of the shape.*

# Mathematics | Grade 4

In Grade 4, instructional time should focus on four critical areas: (1) continuing to develop understanding and fluency with whole number multiplication, and developing understanding of multi-digit whole number division; (2) developing an understanding of addition and subtraction of fractions with like denominators, multiplication of fractions by whole numbers, and division of whole numbers with fractional answers; (3) developing an understanding of area; and (4) understanding that geometric figures can be analyzed and classified using properties such as having parallel sides, perpendicular sides, particular angle measures, and symmetry.

Year Overview

(1) Students use understandings of multiplication to develop fluency with multiplication and division within 100. They apply their understanding of models for multiplication (equal-sized groups, arrays, area models, equal intervals on a number line), place value, and properties of operations, in particular the distributive property, as they develop, discuss, and use efficient, accurate, and generalizable methods to compute products of multi-digit whole numbers. Depending on the numbers and the context, they select and accurately apply appropriate methods to estimate products or mentally calculate products. They develop fluency with efficient procedures, including the standard algorithm, for multiplying whole numbers; understand and explain why the procedures work based on place value and properties of operations; and use them to solve problems. Students apply their understanding of models for division, place value, properties of operations, and the relationship of division to multiplication as they develop, discuss, and use efficient, accurate, and generalizable procedures to find quotients involving multi-digit dividends. They select and accurately apply appropriate methods to estimate quotients and mentally calculate quotients, depending upon the context and the numbers involved.

(2) Students develop understanding of operations with fractions. They apply their understandings of fractions as built from unit fractions, and use fraction models to represent the addition and subtraction of fractions with like denominators. Students use the meaning of fractions and the meaning of multiplication to understand and explain why the procedure for multiplying a fraction by a whole number makes sense. They understand and explain the connection between division and fractions.

(3) Students develop their understanding of area. They understand and apply the area formula for rectangles and also find areas of shapes that can be decomposed into rectangles. They select appropriate units, strategies (e.g., decomposing shapes), and tools for solving problems that involve estimating and measuring area.

(4) Students describe, analyze, compare, and classify two-dimensional shapes. Through building, drawing, and analyzing two-dimensional shapes, students deepen their understanding of properties of two-dimensional objects and the use of them to solve problems involving symmetry.

**Multiplication and division**

1. Find the factor pairs for a given whole number less than or equal to 100; recognize prime numbers as numbers greater than 1 with exactly one factor pair. *Example: The factor pairs of 42 are {42, 1}, {21, 2}, {14, 3}, {7, 6}.*

**Problem solving with the four operations**

2. ❖Solve multistep word problems involving the four operations with whole numbers.
3. ❖Solve problems posed with both whole numbers and fractions. Understand that while quantities in a problem might be described with whole numbers, fractions, or decimals, the operations used to solve the problem depend on the relationships between the quantities regardless of which number representations are involved.
4. Assess the reasonableness of answers using mental computation and estimation strategies including rounding to the nearest 10 or 100.

## Number—Base Ten 4-NBT

**Numbers up to 100,000**

1. Understand that a digit in one place represents ten times what it represents in the place to its right. *For example, 7 in the thousands place represents 10 times as many as than 7 in the hundreds place.*
2. Read, write and compare numbers to 100,000 using base-ten notation, number names, and expanded form.

**Multiplying and dividing in base ten**

3. Understand how the distributive property and the expanded form of a multi-digit number can be used to calculate products of multi-digit numbers.
   a. ❖The product of a one-digit number times a multi-digit number is the sum of the products of the one-digit number with the summands in the expanded form of the multi-digit number. Illustrate this numerically and visually using equations, rectangular arrays, area models, and tape diagrams.
   b. Algorithms for multi-digit multiplication can be derived and explained by writing multi-digit numbers in expanded form and applying the distributive property.
4. Fluently multiply and divide within 100. By end of Grade 4, know from memory products of one-digit numbers where one of the factors is 6, 7, 8, or 9.
5. Mentally calculate products of one-digit numbers and one-digit multiples of 10, 100, and 1000 (e.g., $7 \times 6000$). Mentally calculate whole number quotients with divisors of 10 and 100.
6. Compute products and whole number quotients of two-, three- or four-digit numbers and one-digit numbers, and compute products of two two-digit numbers, using strategies based on place value, the properties of operations, and/or the inverse relationship between multiplication and division; explain the reasoning used.
7. Explain why multiplication and division strategies and algorithms work, using place value and the properties of operations. *Include explanations supported by drawings, equations, or both. A range of reasonably efficient algorithms may be covered, not only the standard algorithms.*
8. Compute products of two-digit numbers using the standard algorithm, and check the result using estimation.
9. Given two whole numbers, find an equation displaying the largest multiple of one which is less than or equal to the other. *For example, given 325 and 7, the equation $325 = 46 \times 7 + 3$ shows the largest multiple of 7 less than or equal to 325.*

## Number—Fractions 4-NF

**Operations on fractions**

1. Understand addition of fractions:
   a. Adding or subtracting fractions with the same denominator means adding or subtracting copies of unit fractions. *For example, $2/3 + 4/3$ is 2 copies of $1/3$ plus 4 copies of $1/3$, or 6 copies of $1/3$ in all, that is $6/3$.*
   b. Sums of related fractions can be computed by replacing one with an equivalent fraction that has the same denominator as the other. *For example, the sum of the related fractions $2/3$ and $1/6$ can be computed by rewriting $2/3$ as $4/6$ and computing $4/6 + 1/6 = 5/6$.*
2. Compute sums and differences of fractions with like denominators, add and subtract related fractions within 1 (e.g., $1/2 + 1/4$, $3/10 + 4/100$, $7/8 - 1/4$), and solve word problems involving these operations.
3. ❖ Understand that the meaning of multiplying a fraction by a whole number comes from interpreting multiplication by a whole number as repeated addition. *For example, $3 \times 2/5 = 6/5$ because $3 \times 2/5 = 2/5 + 2/5 + 2/5 = 6/5$.*

4. Solve word problems that involve multiplication of fractions by whole numbers; represent multiplication of fractions by whole numbers using tape diagrams and area models that explain numerical results.
5. ❖Understand that fractions give meaning to the quotient of any whole number by any non-zero whole number. *For example, $3 \div 4 = 3/4$, because 3/4 multiplied by 4 equals 3. (The division $3 \div 4$ means the number which yields 3 when multiplied by 4.)*
6. Solve word problems that involve non-whole number quotients of whole numbers; represent quotients of whole numbers using tape diagrams and area models that explain numerical results.

**Decimal concepts**

7. Understand that a two-digit decimal is a sum of fractions with denominators 10 and 100. *For example, 0.34 is 3/10 + 4/100.*
8. Use decimals to hundredths to describe parts of wholes; compare and order decimals to hundredths based on meanings of the digits; and write fractions of the form $a/10$ or $a/100$ in decimal notation. *Use > and < symbols to record the results of comparisons.*

## Measurement and Data 4-MD

**The number line and units of measure**

1. Understand that the unit length on a number line (interval from 0 to 1) can be divided into parts of equal fractional length. Draw number line representations of problem situations involving length, height, and distance including fractional or decimal units. *For example, show distances along a race course to tenths of a mile on a number line, by dividing the unit length into 10 equal parts to get parts of length 1/10; the endpoint of the segment of 1/10 length from 0 represents 1/10 of a mile from the starting point of the race. In Grade 4, all numbers lines begin with zero.*

**Perimeter and area**

2. Understand that if a region is decomposed into several disjoint pieces, then the area of the region can be found by adding the areas of the pieces (when these areas are expressed in the same units).
3. ❖Apply the formulas for area of squares and rectangles. Measure and compute whole-square-unit areas of objects and regions enclosed by geometric figures which can be decomposed into rectangles. *Limit to situations requiring products of one-or two-digit numbers.*
4. ❖Find one dimension of a rectangle, given the other dimension and the area or perimeter; find the length of one side of a square, given the area or perimeter. Represent these problems using equations involving a letter for the unknown quantity.

**Angle measurement**

5. Understand what an angle is and how it is measured:
   a. An angle is formed by two rays with a common endpoint.
   b. An angle is measured by reference to a circle with its center at the common endpoint of the rays. The measure of an angle is based on the fraction of the circle between the points where the two rays intersect the circle.
   c. A one-degree angle turns through 1/360 of a circle, where the circle is centered at the common endpoint of its rays; the measure of a given angle is the number of one-degree angles turned with no gaps or overlaps.
6. Measure angles in whole-number degrees using a protractor; sketch angles of specified measure; ❖find the measure of a missing part of an angle, given the measure of the angle and the measure of a part of it, representing these problems with equations involving a letter for the unknown quantity.

**Representing and interpreting data**

7. Make a dot plot to display a data set of measurements in fractions of a unit (1/2, 1/4, 1/8). Solve problems involving addition and subtraction of fractions by using information presented in dot plots. *For example, from a dot plot find and interpret the difference in length between the longest and shortest specimens in an insect collection.*

## Geometry 4-G

**Lines and angles**

1. Draw points, lines, line segments, rays, angles, and perpendicular and parallel lines; identify these in plane figures.
2. Identify right angles, and angles smaller than or greater than a right angle in geometric figures; recognize right triangles.
3. Classify shapes based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of specified size.

**Line symmetry**

4. Understand that a line of symmetry for a geometric figure is a line across the figure such that the figure can be folded along the line into matching parts

5. Identify line-symmetric figures; given a horizontal or vertical line and a drawing that is not a closed figure, complete the drawing to create a figure that is symmetric with respect to the given line.

# Mathematics | Grade 5

In Grade 5, instructional time should focus on four critical areas: (1) developing fluency with addition and subtraction of fractions, developing understanding of the multiplication of fractions and of division of fractions in limited cases (fractions divided by whole numbers and whole numbers divided by unit fractions); (2) developing understanding of and fluency with division of multi-digit whole numbers; (3) developing understanding of and fluency with addition, subtraction, multiplication, and division of decimals; and (4) developing understanding of volume.

(1) Students apply their understanding of fractions and fraction models to represent the addition and subtraction of fractions with unlike denominators as equivalent calculations with like denominators. They develop fluency in calculating sums and differences of fractions, and make reasonable estimates of them. Students also use the meaning of fractions, of multiplication and division, and the inverse relationship between multiplication and division to understand and explain why the procedures for multiplying and dividing fractions make sense. (Note: this is limited to the case of dividing fractions by whole numbers and whole numbers by unit fractions.)

(2) Students develop fluency with division of whole numbers; understand why procedures work based on the meaning of base-ten notation and properties of operations; and use these procedures to solve problems. Based on the context of a problem situation, they select the most useful form of the quotient for the answer and interpret it appropriately.

(3) Students apply their understandings of models for decimals, decimal notation, and properties of operations to compute sums and differences of finite decimals. They develop fluency in these computations, and make reasonable estimates of their results. Students use the relationship between decimals and fractions, as well as the relationship between finite decimals and whole numbers (i.e., a finite decimal multiplied by an appropriate power of 10 is a whole number), to understand and explain why the procedures for multiplying and dividing finite decimals make sense. They compute products and quotients of finite decimals efficiently and accurately.

(4) Students recognize volume as an attribute of three-dimensional space. They understand that volume can be quantified by finding the total number of same-size units of volume required to fill the space without gaps or overlaps. They understand that a 1-unit by 1-unit by 1-unit cube is the standard unit for measuring volume. They select appropriate units, strategies, and tools for solving problems that involve estimating and measuring volume. They decompose three-dimensional shapes and find volumes of right rectangular prisms by viewing them as decomposed into layers of arrays of cubes. They measure necessary attributes of shapes in order to determine volumes to solve problems.

**Whole numbers in base ten**

1. Compute quotients of two-, three-, and four-digit whole numbers and two-digit whole numbers using strategies based on place value, the properties of operations, and/or the inverse relationship between multiplication and division; explain the reasoning used.
2. Explain why division strategies and algorithms work, using place value and the properties of operations. *Include explanations supported by drawings, equations, or both. A range of reasonably efficient algorithms may be covered, not only the standard algorithm.*
3. Use the standard algorithm to compute quotients of two-, three- and four-digit whole numbers and two-digit whole numbers, expressing the results as an equation (e.g., $145 = 11 \times 13 + 2$ or $120 \div 7 = 17\ 1/7$).
4. Fluently add, subtract and multiply whole numbers using the standard algorithm for each operation.

**Decimal concepts**

5. Read, write, and compare numbers expressed as decimals. Understand that a digit in one place represents ten times what it represents in the place to its right. *For example, 7 in the hundredths place represents 10 times as many as 7 in the thousandths place.*
6. Round decimals (to hundredths) to the nearest whole number.
7. Write fractions in decimal notation for fractions with denominators 2, 4, 5, 8, 10, and 100.

**Operations on decimals**

8. Understand that in adding or subtracting finite decimals, one adds or subtracts like units (tenths and tenths, hundredths and hundredths, etc.) and sometimes it is necessary to compose or decompose a higher value unit.
9. Fluently find 0.1 more than a number and less than a number; 0.01 more than a number and less than a number; and 0.001 more than a number and less than a number, for numbers expressed as finite decimals.
10. Compute sums and differences of finite decimals by expressing the decimals as fractions and adding the fractions. *For example,* $0.05 + 0.91 = 5/100 + 91/100 = 96/100$ *or* $0.96$.
11. Compute sums, differences, products, and quotients of finite decimals using strategies based on place value, the properties of operations, and/or the inverse relationships between addition and subtraction and between multiplication and division; explain the reasoning used. *For example, transform* $1.5 \div 0.3$ *into* $15 \div 3 = 5$.
12. Explain why strategies and algorithms for computations with finite decimals work. *Include explanations supported by drawings, equations, or both. A range of reasonably efficient algorithms may be covered, not only the standard algorithm.*
13. Use the standard algorithm for each of the four operations on decimals (to hundredths).
14. Solve word problems involving operations on decimals.

## Number—Fractions 5-NF

**Fraction equivalence**

1. ❖Understand fraction equivalence:
   a. Multiplying the numerator and denominator of a fraction by the same nonzero whole number produces an equivalent fraction. *For example,* $2/3 = (2 \times 4)/(3 \times 4) = 8/12$. *(1/3 is 4 copies of 1/12, so 2/3 is 8 copies of 1/12.)*
   b. Equivalent fractions correspond to the same point on a number line. *In Grade 5, all numbers lines begin with zero.*
   c. When the numerators of equivalent fractions are divided by their denominators, the resulting quotients are the same.
2. Identify pairs of equivalent fractions; given two fractions with unlike denominators, find two fractions with the same denominator and equivalent to each.
3. Compare and order fractions with like or unlike denominators, e.g., by finding equivalent fractions with the same denominator, and describe the sizes of fractional quantities from a context with reference to the context. *Compare using the fractions themselves, tape diagrams or number line representations, and area models.*

**Operations on fractions**

4. Understand that sums and differences of fractions with unlike denominators can be computed by replacing each with an equivalent fraction so that the resulting fractions have the same denominator. *For example,* $2/3 + 5/4 = 8/12 + 15/12 = 23/12$.
5. Compute sums and differences of fractions with like or unlike denominators, and solve word problems involving addition and subtraction of fractions. Estimate fraction sums and differences to assess the reasonableness of results.
6. ❖ Understand that multiplying a fraction by $a/b$ means taking $a$ parts of a decomposition of the fraction into $b$ equal parts. *For example, to multiply* $2/3 \times 4/5 = 8/15$, *one may decompose a whole of size 4/5 into 3 equal parts; each part has size 4/15. Two*

*of these parts then make 8/15, so 2/3 × 4/5 = 8/15. (In general, a/b × p/q = ap/bq.) This standard includes multiplication of a whole number by a fraction, by writing the whole number as fraction with denominator 1.*

7. Understand that the area of a rectangle with side lengths $a/b$ and $c/d$ is the product $a/b \times p/q$. This extends the area formula for rectangles to fractional side lengths, and also allows products of fractions to be represented visually as areas of rectangles.
8. *Explain and justify the properties of operations with fractions, e.g., by using equations, number line representations, area models, and story contexts.
9. Understand division of unit fractions by whole numbers and division of whole numbers by unit fractions:
   a. Dividing a unit fraction 1/b by a whole number a results in a smaller unit fraction 1/a × b. *For example, 1/3 ÷ 2 = 1/6 because when 1/3 is divided into 2 equal parts, the size of each part is 1/6; a third of a pound of cheese shared between two people will give each person a sixth of a pound. (Using the inverse relationship between multiplication and division: 1/3 ÷ 2 = 1/6 because 1/6 × 2 = 1/3.)*
   b. Dividing a whole number a by a unit fraction 1/b results in a greater whole number a × b. *For example, 2 ÷ 1/3 = 6 because 6 is the number of 1/3s in 2; two pounds of cheese will make six portions of a third of a pound each. (Using the inverse relationship between multiplication and division: 2 ÷ 1/3 = 6 because 6 × 1/3 = 2.)*
10. Calculate products of fractions, and quotients of unit fractions and nonzero whole numbers (with either as divisor), and solve word problems involving these operations. Represent these operations using equations, area models and length models.
11. Understand that a mixed number such as 3 2/5 represents the sum of a whole number and a fraction less than one. Because a whole number can be represented as a fraction (3 = 3/1), and the sum of two fractions is also a fraction, a mixed number also represents a fraction (3 2/5 = 3 + 2/5 = 15/5 + 2/5 = 17/5). Write fractions as equivalent mixed numbers and vice versa.

## Measurement and Data

5-MD

**Units of measure**

1. Understand that quantities expressed in like units can be added or subtracted giving a sum or difference with the same unit; different quantities may be multiplied to obtain a new kind of quantity (e.g., as when two lengths are multiplied to compute an area, or when an area and a length are multiplied to compute a volume).
2. Understand that when measuring a quantity, if a smaller unit is used, more units must be iterated to measure the quantity in those units.
3. Convert among different-sized standard measurement units within a given measurement system (e.g., feet to yards, centimeters to meters) and use conversion in solving multi-step word problems.

**Volume**

4. Understand concepts of volume measurement:
   a. A cube with side length 1 unit (a unit cube) is said to have "one cubic unit" of volume, and can be used to measure volume.
   b. The volume of a right rectangular prism with whole-unit side lengths can be found by packing it with unit cubes and using multiplication to count their number. *For example, decomposing a right rectangular prism 3 length units wide by 5 units deep by 2 units tall shows that its volume is 3 × 5 × 2 cubic units. The base of the prism has area 3 × 5 square units, so the volume can also be expressed as the height times the area of the base.*
   c. When measuring a volume, if a smaller unit is used, more units must be iterated to measure the volume in those units.
   d. If a solid figure is decomposed into several disjoint pieces, then the volume enclosed by the figure can be found by adding the volumes of the pieces (when these volumes are expressed in the same units).
5. Decompose right rectangular prisms into layers of arrays of cubes; determine and compare volumes of right rectangular prisms, and objects well described as right rectangular prisms, by counting cubic units (using $cm^3$, $m^3$, $in^3$, $ft^3$, and improvised units).

**Representing and interpreting data**

6. Make a dot plot to display a data set of measurements in fractions of a unit (1/2, 1/4, 1/8). Use operations on fractions for this grade to solve problems involving information presented in dot plots. *For example, given different measurements of liquid in identical beakers, find the amount of liquid each beaker would contain if the total amount in all the beakers were redistributed equally.*

## Geometry

5-G

**Coordinates**

1. Understand that a pair of perpendicular number lines, called axes, defines a coordinate system.
   a. Their intersection is called the origin, usually arranged to coincide with the 0 on each line.
   b. A given point in the plane can be located by using an ordered pair of numbers, called its coordinates. The first number indicates how far to travel from the origin in the direction of one axis, the second number indicates how far to travel in the direction of the second axis.
   c. To avoid ambiguity, conventions dictate that the names of the two axes and the coordinates correspond (e.g., $x$-axis and $x$-coordinate, $y$-axis and $y$-coordinate).
2. Graph points in the first quadrant of the coordinate plane, and identify the coordinates of graphed points. Where ordered pairs arise in a problem situation, interpret the coordinate values in the context of the situation.

**Plane figures**

3. Understand that properties belonging to a category of plane figures also belong to all subcategories of that category. *For example, all rectangles have four right angles and squares are rectangles, so all squares have four right angles.*
4. Classify plane figures in a hierarchy based on properties.

# Mathematics | Grade 6

In Grade 6, instructional time should focus on four critical areas: (1) connecting ratio and rate to whole number multiplication and division; (2) developing understanding of and fluency with division of fractions and developing fluency with multiplication of fractions; (3) developing understanding of and using formulas to determine areas of two-dimensional shapes and distinguishing between volume and surface area of three-dimensional shapes; and (4) writing, interpreting, and using expressions and equations.

(1) Students use reasoning about multiplication and division with quantities to solve ratio and rate problems. By viewing equivalent ratios and rates as deriving from, and extending, pairs of rows (or columns) in the multiplication table, and by analyzing simple drawings that indicate the relative size of quantities, students extend whole number multiplication and division to ratios and rates. Thus students expand their repertoires of problems in which multiplication and division can be used to solve problems, and they build on their understanding of fractions to understand ratios. Students solve a wide variety of problems involving ratios and rates.

(2) Students use the meaning of fractions, the meanings of multiplication and division, and the inverse relationship between multiplication and division to understand and explain why the procedures for dividing fractions make sense. Students are able to add, subtract, multiply, and divide fractions fluently, and use these operations to solve problems, including multi-step problems and problems involving measurement.

(3) Students reason about relationships among shapes to determine area and surface area. They find areas of right triangles, other triangles, and special quadrilaterals by decomposing these shapes, rearranging or removing pieces, and relating the shapes to rectangles. Using these methods, students discuss, develop, and justify formulas for areas of triangles and parallelograms. Students find areas of polygons and surface areas of prisms and pyramids by decomposition into pieces whose area they can determine.

(4) Students write mathematical expressions and equations that correspond to given situations, they evaluate expressions, and they use expressions and formulas to solve problems. Students understand that a variable is a letter standing for a number, where the number is unknown, or where, for the purpose at hand, it can be any number in the domain of interest. Students understand that expressions in different forms can be equivalent, and they use the laws of arithmetic to rewrite expressions to represent a total quantity in a different way (such as to represent it more compactly or to feature different information). Students know that the solutions of an equation are the values of the variables that make the equation true. Students use properties of operations and the idea of maintaining the equality of both sides of an equation to solve simple one-step equations. Students construct and analyze tables, such as tables of quantities that are in equivalent ratios, and they use equations (such as $3x = y$) to describe relationships in a table.

Having represented and analyzed data in Grades K–5, students in Grade 6 begin a serious engagement with statistics. The study of variability in data distinguishes statistics from mathematics. Students beginning their study of variability must first recognize statistical questions as those that anticipate variability in the answers. From this conceptual beginning, they learn to describe and summarize distributions of data—an activity that goes beyond merely computing summary statistics to include assessing the shape of a distribution and considering other issues as described in the standards.

**Ratios**

1. Understand the concept of a ratio: Two quantities are said to be in a ratio of $a$ to $b$ when for every $a$ units of the first quantity there are $b$ units of the second. *For example, in a flock of birds, the ratio of wings to beaks might be 2 to 1; this ratio is also written 2:1. In Grade 6, limit to ratios of whole numbers.*
2. Make tables of equivalent ratios relating quantities with whole-number measurements, find missing values in the tables, and plot the pairs of values on the coordinate plane.
3. Solve for an unknown quantity in a problem involving two equal ratios.
4. Describe categorical data sets using ratios (e.g., for every vote candidate A received, candidate C received nearly three votes; the ratio of type O blood donors to type B blood donors was 9:2).

**Unit rates**

5. Understand that for a ratio $a:b$, the corresponding unit rate is $^a/_b$. If there are $a$ units of the first quantity for every $b$ units of the second, where $b \neq 0$, then there are $^a/_b$ units of the first quantity for 1 unit of the second. *For example, if a recipe has a ratio of 3 cups of flour to 4 cups of sugar, then there is 3/4 cup of flour for each cup of sugar.*
6. ❖Solve unit rate problems including unit pricing and constant speed, including reasoning with equations such as $d = r \times t$, $r = {}^d/_t$, $t = d \div r$.

## The Number System 6-NS

**Operations**

1. Understand that the properties of operations apply to, and can be used with, addition and multiplication of fractions.
2. Understand that division of fractions is defined by viewing a quotient as the solution for an unknown-factor multiplication problem. *For example, $(2/3) \div (5/7) = 14/15$ because $(5/7) \times (14/15) = (2/3)$.*
3. Solve word problems requiring arithmetic with fractions, using the properties of operations and converting between forms as appropriate; estimate to check reasonableness of answers.
4. Fluently divide whole numbers using the standard algorithm.

**The system of rational numbers**

5. Understand that a number is a point on the number line.
6. Understand that some quantities have opposite directions, such as elevation above and below sea level or money received and spent. These quantities can be described using positive and negative numbers.
7. Understand that number lines familiar from previous grades can be extended to represent negative numbers to the left of zero. *Number lines can also be vertically oriented, as when a coordinate system is formed. Then the conventional terms "to the right of 0" and "to the left of 0" conventionally become "above 0" and "below 0."*
   a. Two different numbers, such as 7 and –7, that are equidistant from zero on a number line are said to be opposites of one another. The opposite of the opposite of a number is the number itself, e.g., $-(-3) = 3$. The opposite of 0 is 0.
   b. The absolute value of a number $q$, written $|q|$, is its distance from zero, and is always positive or zero.
   c. Fractions and their opposites form a system of numbers called the rational numbers, represented by points on a number line. Whole numbers and their opposites form the integers, which are contained in the rational numbers.
   d. Previous ways of comparing positive numbers can be extended to the rational numbers. The statement $p > q$ means that $p$ is located to the right of $q$ on a number line, while $p < q$ means that $p$ is located to the left of $q$ on a number line. Comparisons can also be made by reasoning appropriately about signed quantities (e.g, $-3 > -7$ makes sense because –3°C is a higher temperature than –7°C). The way two numbers compare does not always agree with the way their absolute values compare; for example, $-3 > -7$, but $|-3| < |-7|$.
8. Find and position rational numbers, including integers, on a number line.
9. Use rational numbers to describe quantities such as elevation, temperature, account balance and so on. Compare these quantities, recording the results of comparisons using > and < symbols.
10. Graph points and identify coordinates of points on the coordinate plane in all four quadrants. Where ordered pairs arise in a problem situation, interpret the coordinate values in the context of the situation.

**Expressions**

1. Understand that an expression records operations with numbers or with letters standing for numbers. *For example, the expression $2 \cdot (8 + 7)$ records adding 8 and 7 then multiplying by 2; the expression $5 - y$ records subtracting y from 5. Focus on the operations of addition, subtraction, multiplication and division, with some attention to square or cube roots.*
2. Understand the use of variables in expressions and algebraic conventions:
   a. A letter is used to stand for a number in an expression in cases where the number is unknown, or where, for the purpose at hand, it can be any number in a domain of interest. Such a letter is called a variable.
   b. If a variable appears in an expression more than once (e.g., as in $t + 3t$), that variable is understood to refer to the same number in each instance.
   c. The multiplication symbol can be omitted when writing products of two or more variables or of a number and a variable. *For example, the expressions xy and 2a indicate $x \times y$ and $2 \times a$, respectively.*
3. Describe the structure and elements of simple expressions using correct terminology (sum, term, product, factor, quotient, coefficient); describe an expression by viewing one or more of its parts as a single entity. *For example, describe the expression $2 \cdot (8 + 7)$ as a product of two factors, by viewing $(8 + 7)$ as a single entity. The second factor is itself a sum of two terms.*
4. Understand and generate equivalent expressions:
   a. Understand that two expressions are equivalent if they name the same number regardless of which numbers the variables in them stand for. *For example, the expressions $x + 3$ and $4x$ are not equivalent, even though they happen to name the same number in the case when x stands for 1.*
   b. Understand that applying the laws of arithmetic to an expression results in an equivalent expression. *For example, applying the distributive law to the expression $3 \cdot (2 + x)$ leads to the equivalent expression $6 + 3x$. Applying the distributive law to $y + y + y$ leads to the equivalent expression $y \times (1 + 1 + 1)$, i.e., $y \times 3$ and then the commutative law of multiplication leads to the equivalent expression $3y$.*
   c. Generate equivalent expressions to reinterpret the meaning of an expression. *For example, $2t + 3t$ records the addition of twice a quantity to three times itself; applying the distributive law leads to the equivalent expression $5t$, so that the original expression can be reinterpreted as recording five times the quantity.*

**Quantitative relationships and the algebraic approach to problems**

5. Understand that an equation is a statement that two expressions are equal, and a solution to an equation is a replacement value of the variable (or replacement values for all the variables if there is more than one) that makes the equation true.
6. Using the idea of maintaining equality between both sides of the equation, solve equations of the form $x + p = q$ and $px = q$ for cases in which $p$, $q$ and $x$ are all nonnegative rational numbers.
7. Choose variables to represent quantities in a word problem, and construct simple expressions or equations to solve the problem by reasoning about the quantities.
8. Understand that a variable can be used to represent a quantity that can change, often in relationship to another changing quantity, and an equation can express one quantity, thought of as the dependent variable, in terms of other quantities, thought of as the independent variables; represent a relationship between two quantities using equations, graphs, and tables; translate between any two of these representations. *For example, describe the terms in a sequence $t = 3, 6, 9, 12, \ldots$ of multiples of 3 by writing the equation $t = 3n$ for $n = 1, 2, 3, 4, \ldots$.*

## Geometry 6-G

**Properties of area, surface area, and volume**

1. Understand that plane figures can be decomposed, reassembled, and completed into new figures; use this technique to derive area formulas.
2. Find the areas enclosed by right triangles, other triangles, special quadrilaterals, and polygons (by composing into rectangles or decomposing into triangles and other shapes).
3. Understand that three-dimensional figures can be formed by joining rectangles and triangles along their edges to enclose a solid region with no gaps or overlaps. The surface area is the sum of the areas of the enclosing rectangles and triangles.
4. Find the surface area of cubes, prisms and pyramids (include the use of nets to represent these figures).
5. Solve problems involving area, volume and surface area of objects.
6. Give examples of right rectangular prisms with the same surface area and different volumes, and with the same volume and different surface areas.

7. ❖Use exponents and symbols for square roots and cube roots to express the area of a square and volume of a cube in terms of their side lengths, and to express their side lengths in terms of their area or volume.

## Statistics and Probability 6-SP

**Variability and measures of center**

1. Understand that a statistical question is one that anticipates variability in the data related to the question and accounts for it in the answers. *For example, "How old am I?" is not a statistical question , but "How old are the students in my school?" is a statistical question because one anticipates variability in students' ages.*
2. Understand that a set of data generated by answers to a statistical question typically shows variability—not all of the values are the same—and yet often the values show an overall pattern, often with a tendency to cluster.
   a. A measure of center for a numerical data set summarizes all of its values using a single number. The median is a measure of center in the sense that approximately half the data values are less than the median, while approximately half are greater. The mean is a measure of center in the sense that it is the value that each data point would take on if the total of the data values were redistributed fairly, and in the sense that it is the balance point of a data distribution shown on a dot plot.
   b. A measure of variation for a numerical data set describes how its values vary using a single number. The interquartile range and the mean absolute deviation are both measures of variation.

**Summarizing and describing distributions**

3. Display numerical data in plots on a number line, including dot plots, histograms, and box plots.
4. Summarize numerical data sets, such as by:
   a. Reporting the number of observations.
   b. Describing the nature of the variable, including how it was measured and its units of measurement. *Data sets can include fractional values at this grade but not negative values.*
   c. Describing center and variation, as well as describing any overall pattern and any striking deviations from the overall pattern.
5. Relate the choice of the median or mean as a measure of center to the shape of the data distribution being described and the context in which it is being used. Do the same for the choice of interquartile range or mean average deviation as a measure of variation. *For example, why are housing prices often summarized by reporting the median selling price, while students' assigned grades are often based on mean homework scores?*

# Mathematics | Grade 7

In Grade 7, instructional time should focus on four critical areas: (1) developing understanding of and applying proportional relationships; (2) developing understanding of operations with rational numbers and solving linear equations; (3) analyzing two- and three-dimensional space and figures using distance, angle, similarity, and congruence; and (4) drawing inferences about populations based on samples.

(1) Students extend their understanding of ratios and develop understanding of proportionality to solve single- and multi-step problems. Students use their understanding of ratios and proportionality to solve a wide variety of percent problems, including those involving discounts, interest, taxes, tips, and percent increase or decrease. Students solve problems about similar objects (including geometric figures) by using scale factors that relate corresponding lengths between the objects or by using the fact that relationships of lengths within an object are preserved in similar objects. Students graph proportional relationships and understand the unit rate informally as a measure of the steepness of the related line, called the slope. They distinguish proportional relationships from other relationships.

(2) Students develop a unified understanding of number, recognizing fractions, decimals, and percents as different representations of rational numbers. Students extend addition, subtraction, multiplication, and division and their properties to all rational numbers, including integers and numbers represented by complex fractions and negative fractions. By applying the laws of arithmetic, and by viewing negative numbers in terms of everyday contexts (e.g., amounts owed or temperatures below zero), students explain why the rules for adding, subtracting, multiplying, and dividing with negative numbers make sense. They use the arithmetic of rational numbers as they formulate and solve linear equations in one variable and use these equations to solve problems.

(3) Students use ideas about distance and angles, how they behave under dilations, translations, rotations and reflections, and ideas about congruence and similarity to describe and analyze figures and situations in two- and three-dimensional space and to solve problems, including multi-step problems. Students prove that various configurations of lines give rise to similar triangles because of the angles created when a transversal cuts parallel lines. Students apply this reasoning about similar triangles to solve problems, such as finding heights and distances. Students see the plausibility of the formulas for the circumference and area of a circle. For example, in the case of area, they may do so by reasoning about how lengths and areas scale in similar figures or by decomposing a circle or circular region and rearranging the pieces.

(4) Students build on their previous work with single data distributions to compare two data distributions and address questions about differences between populations. They begin informal work with random sampling to generate data sets and learn about the importance of representative samples for drawing inferences.

**Analyzing proportional relationships**

1. Form ratios of nonnegative rational numbers and compute corresponding unit rates. *For example, a person might walk ½ mile in each ¼ hour; the unit rate for this ratio is (1/2)/(1/4) miles per hour, equivalently 2 miles per hour. Include ratios of lengths, areas and other quantities, including when quantities being compared are measured in different units.*
2. Recognize situations in which two quantities covary and have a constant ratio. (The quantities are then said to be in a proportional relationship and the unit rate is called the constant of proportionality.) Decide whether two quantities that covary are in a proportional relationship, e.g., by testing for equivalent ratios or graphing on a coordinate plane.
3. Compute unit rates and solve proportional relationship problems in everyday contexts, such as shopping, cooking, carpentry, party planning, etc. Represent proportional relationships by equations that express how the quantities are related via the constant of proportionality or unit rate. *For example, total cost, t, is proportional to the number, n, purchased at a constant price, p; this relationship can be expressed as* $t = pn$.
4. Plot proportional relationships on a coordinate plane where each axis represents one of the two quantities involved, observe that the graph is a straight line through the origin, and find unit rates from a graph. Explain what a point $(x, y)$ means in terms of the situation, with special attention to the points (0, 0) and $(1, r)$ where $r$ is the unit rate.
5. Compare tables, graphs, formulas, diagrams, and verbal descriptions that represent or partially represent proportional relationships; explain correspondences among the representations including how the unit rate is shown in each.

**Percent**

6. Understand that percentages are rates per 100. For example, 30% of a quantity means $^{30}/_{100}$ times the quantity. A percentage can be a complex fraction, as in 3.75% = $^{3.75}/_{100}$.
7. Find a percentage of a quantity; solve problems involving finding the whole given a part and the percentage.
8. Solve multistep percent problems. *Examples: simple interest, tax, markups and markdowns, gratuities and commissions, fees, percent increase and decrease, percent error, expressing monthly rent as a percentage of take-home pay.*

## The Number System 7-NS

**The system of rational numbers**

1. Understand that the rules for manipulating fractions extend to complex fractions.
2. Understand and perform addition and subtraction with rational numbers:
   a. Understand that on a number line, the sum $p + q$ is the number located a distance $|q|$ from $p$, to the right of $p$ if $q$ is positive and to the left of $p$ if $q$ is negative. A number and its opposite are additive inverses (i.e., their sum is zero).
   b. Compute sums of signed numbers using the laws of arithmetic. *For example,* $7 + (-3) = 4$ *because* $7 + (-3) = (4 + 3) + (-3) = 4 + [3 + (-3)] = 4 + [0] = 4$.
   c. Understand that subtraction of rational numbers is defined by viewing a difference as the solution of an unknown-addend addition problem. Subtraction of a rational number gives the same answer as adding its additive inverse.
   d. Explain and justify rules for adding and subtracting rational numbers, using a number line and practical contexts. *For example, relate* $r + (-s) = r - s$ *to a bank transaction; explain why* $p - (q + r) = p - q - r$.
   e. Understand that the additive inverse of a sum is the sum of the additive inverses, that is $-(p + q) = -p + -q$. *For example,* $-(6 + -2) = (-6) + 2$ *because* $[6 + (-2)] + [(-6) + 2] = [6 + (-6)] + [(-2) + 2] = [0] + [0] = 0$.
3. Understand and perform multiplication and division with rational numbers:
   a. Understand that the extension of multiplication from fractions to rational numbers is determined by the requirement that multiplication and addition satisfy the laws of arithmetic, particularly the distributive law, leading to products such as $(-1)(-1) = 1$ and the rules for multiplying signed numbers.
   b. Understand that integers can be divided, provided that the divisor is not zero, and every quotient of integers (with non-zero divisor) is a rational number. If $p/q$ is a rational number, then $-(p/q) = (-p)/q = p/(-q)$.
   c. Calculate products and quotients of rational numbers, and use multiplication and division to solve word problems. *Include signed quantities.*

**The system of real numbers**

4. Understand that there are numbers that are not rational numbers, called irrational numbers, e.g., $\pi$ and $\sqrt{2}$. Together the rational and irrational numbers form the real number system. In school mathematics, the real numbers are assumed to satisfy the laws of arithmetic.

## Expressions and Equations 7-EE

**Expressions**

1. Interpret numerical expressions at a level necessary to calculate their value using a calculator or spreadsheet. For expressions with variables, use and interpret conventions of algebraic notation, such as $y/2$ is $y \div 2$ or $1/2 \times y$; $(3 \pm y)/5$ is $(3 \pm y) \div 5$ or $1/5 \times (3 \pm y)$; $a^2$ is $a \times a$, $a^3$ is $a \times a \times a$, $a^2b$ is $a \times a \times b$.
2. Generate equivalent expressions from a given expression using the laws of arithmetic and conventions of algebraic notation. Include:
   a. Adding and subtracting linear expressions, as in $(2x + 3) + x + (2 - x) = 2x + 5$.
   b. Factoring, as in $4x + 4y = 4(x + y)$ or $5x + 7x + 10y + 14y = 12x + 24y = 12(x + 2y)$.
   c. Simplifying, as in $-2(3x - 5) + 4x = 10 - 2x$ or $x/3 + (x - 2)/4 = 7x/12 - 1/2$.

**Quantitative relationships and the algebraic approach to problems**

3. Choose variables to represent quantities in a word problem, and construct simple equations to solve the problem by reasoning about the quantities.
   a. Solve word problems leading to equations of the form $px + q = r$ and $p(x + q) = r$, where $p$, $q$, and $r$ are nonnegative rational numbers and the solution is a nonnegative rational number. Fluently solve equations of these forms, e.g., by undoing the operations involved in producing the expression on the left.
   b. Solve the same word problem arithmetically and algebraically. *For example, "J. has 4 packages of balloons and 5 single balloons. In all, he has 21 balloons. How many balloons are in a package?" Solve this problem arithmetically (using a sequence of operations on the given numbers), and also solve it by using a variable to stand for the number of balloons in a package, constructing an equation such as $4b + 5 = 21$ to describe the situation then solving the equation.*
   c. Understand that rewriting an expression in different forms in a problem context can shed light on the problem and how the quantities in it are related. *For example, $P + 0.05P = 1.05P$ means that "increase by 5%" is the same as "multiply by 1.05."*

## Geometry

7-G

**Congruence and similarity**

1. Verify experimentally the fact that a rigid motion (a sequence of rotations, reflections, and translations) preserves distance and angle, e.g., by using physical models, transparencies, or dynamic geometry software:
   a. Lines are taken to lines, and line segments to line segments of the same length.
   b. Angles are taken to angles of the same measure.
   c. Parallel lines are taken to parallel lines.
2. Understand the meaning of congruence: a plane figure is congruent to another if the second can be obtained from the first by a rigid motion.
3. Verify experimentally that a dilation with scale factor $k$ preserves lines and angle measure, but takes a line segment of length $L$ to a line segment of length $kL$.
4. Understand the meaning of similarity: a plane figure is similar to another if the second can be obtained from the first by a similarity transformation (a rigid motion followed by a dilation).
5. Solve problems involving similar figures and scale drawings. *Include computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.*
6. Use informal arguments involving approximation by lines, squares, and cubes to see that a similarity transformation with a scale factor of $k$ leaves angle measures unchanged, changes lengths by a factor of $k$, changes areas by a factor of $k^2$, and changes volumes by a factor of $k^3$.
7. Know the formulas relating the area, radius and circumference of a circle and solve problems requiring the use of these formulas; give an informal derivation of the relationship between the circumference and area of a circle.

**Angles**

8. Justify facts about the angle sum of triangles, exterior angles, and alternate interior angles created when parallel lines are cut by a transversal, e.g., by using physical models, transparencies, or dynamic geometry software to make rigid motions and give informal arguments. *For example, arrange three copies of the same triangle so that the three angles appear to form a line, and give an argument in terms of transversals why this is so.*
9. Use facts about supplementary, complementary, vertical, and adjacent angles in a multi-step problem to write and solve simple equations for an unknown angle in a figure.

**Situations involving randomness**

1. Simulate situations involving randomness using random numbers generated by a calculator or a spreadsheet or taken from a table. *For example, if you guess at all ten true/false questions on a quiz, how likely are you to get at least seven answers correct?*
2. Use proportional reasoning to predict relative frequencies of outcomes for situations involving randomness, but for which a theoretical answer can be determined. For *example, when rolling a number cube 600 times, one would predict that a 3 or 6 would be rolled roughly 200 times, but probably not exactly 200 times. How far off might your prediction be? Use technology to generate multiple samples to approximate a distribution of sample proportions. Repeat the process for smaller sample sizes.*

**Random sampling to draw inferences about a population**

3. Understand that statistics can be used to gain information about a population by examining a sample of the population; generalizations about a population from a sample are valid only if the sample is representative of that population. Understand that random sampling tends to produce representative samples and support valid inferences.
4. Understand the importance of measures of variation in sample quantities (like means or proportions) in reasoning about how well a sample quantity estimates or predicts the corresponding population quantity.
5. Use data from a random sample to draw inferences about a population with an unknown characteristic of interest. Generate multiple samples (or simulated samples) of the same size to gauge the variation in estimates or predictions. *For example, estimate the mean word length in a book by randomly sampling words from the book; predict the winner of a school election based on randomly sampled survey data. Gauge how far off the estimate or prediction might be.*

**Comparative inferences about two populations**

6. Informally assess the degree of visual overlap of two numerical data distributions with similar variabilities, measuring the difference between the centers by expressing it as a multiple of a measure of variability. *For example, the mean height of players on the basketball team is 10 cm greater than the mean height of players on the soccer team, about twice the variability (mean average deviation) on either team; on a dot plot, the separation between the two distributions of heights is noticeable.*
7. Use measures of center and measures of variability for numerical data from uniform random samples to draw informal comparative inferences about two populations. *For example, decide whether the words in a chapter of a seventh-grade book are generally longer than the words in a chapter of a sixth-grade book.*

# Mathematics | Grade 8

In Grade 8, instructional time should focus on three critical areas: (1) solving linear equations and systems of linear equations; (2) grasping the concept of a function and using functions to describe quantitative relationships; (3) understanding and applying the Pythagorean Theorem.

(1) Students use linear equations, and systems of linear equations to represent, analyze, and solve a variety of problems. Students recognize proportions ($y/x = m$ or $y = mx$) as a special case of linear equations, $y = mx + b$, understanding that the constant of proportionality ($m$) is the slope and the graphs are lines through the origin. They understand that the slope ($m$) of a line is a constant rate of change, so that if the input or $x$-coordinate changes by an amount $A$, the output or $y$-coordinate changes by the amount $mA$. Students also formulate and solve linear equations in one variable and use these equations to solve problems. Students also use a linear equation to describe the association between two quantities in a data set (such as arm span vs. height for students in a classroom). At this grade, fitting the model, and assessing its fit to the data are done informally. Interpreting the model in the context of the data requires students to express a relationship between the two quantities in question.

Students strategically choose and efficiently implement procedures to solve linear equations in one variable, understanding that when they use the properties of equality and the concept of logical equivalence, they maintain the solutions of the original equation. Students solve systems of two linear equations in two variables and relate the systems to pairs of lines in the plane; these intersect, are parallel, or are the same line. Students use linear equations, systems of linear equations, linear functions, and their understanding of slope of a line to analyze situations and solve problems.

(2) Students grasp the concept of a function as a rule that assigns to each element of its domain exactly one element of its range. They use function notation and understand that functions describe situations where one quantity determines another. They can translate among verbal, tabular, graphical, and algebraic representations of functions (noting that tabular and graphical representations are usually only partial representations), and they describe how aspects of the function are reflected in the different representations.

(3) Students understand the statement of the Pythagorean Theorem and its converse, and can explain why the Pythagorean Theorem is valid, for example, by decomposing a square in two different ways. They apply the Pythagorean Theorem to find distances between points on the coordinate plane, to find lengths, and to analyze polygons.

**The system of real numbers**

1. Understand informally that every number on a number line has a decimal expansion, which can be found for rational numbers using long division. Rational numbers are those with repeating decimal expansions (this includes finite decimals which have an expansion that ends in a sequence of zeros).
2. Informally explain why $\sqrt{2}$ is irrational.
3. Use rational approximations (including those obtained from truncating decimal expansions) to compare the size of irrational numbers, locate them approximately on a number line, and estimate the value of expressions (e.g., $\pi^2$). *For example, show that the square root of 2 is between 1 and 2, then between 1.4 and 1.5, and explain how to continue on to get better approximations.*

## Expressions and Equations 8-EE

**Linear equations in one variable**

1. Understand that a linear equation in one variable might have one solution, infinitely many solutions, or no solutions. Which of these possibilities is the case can be determined by successively transforming the given equation into simpler forms, until an equivalent equation of the form $x = a$, $a = a$, or $a = b$ results (where $a$ and $b$ are different numbers).
2. Solve linear equations with rational number coefficients, including equations that require expanding expressions using the distributive law and collecting like terms.

**Linear equations in two variables**

3. Understand that the slope of a non-vertical line in the coordinate plane has the same value for any two distinct points used to compute it. This can be seen using similar triangles.
4. Understand that two lines with well-defined slopes are parallel if and only if their slopes are equal.
5. Understand that the graph of a linear equation in two variables is a line, the set of pairs of numbers satisfying the equation. If the equation is in the form $y = mx + b$, the graph can be obtained by shifting the graph of $y = mx$ by $b$ units (upwards if $b$ is positive, downwards if $b$ is negative). The slope of the line is $m$.
6. Understand that a proportional relationship between two variable quantities $y$ and $x$ can be represented by the equation $y = mx$. The constant $m$ is the unit rate, and tells how much of $y$ per unit of $x$.
7. Graph proportional relationships and relationships defined by a linear equation; find the slope and interpret the slope in context.
8. Compare two different proportional relationships represented in different ways. *For example, compare a distance-time graph to a distance-time equation to determine which of two moving objects has greater speed.*

**Systems of linear equations**

9. Understand that solutions to a system of two linear equations in two variables correspond to points of intersection of their graphs, because points of intersection satisfy both equations simultaneously.
10. Solve systems of two linear equations in two variables algebraically, and estimate solutions by graphing the equations. Solve simple cases by inspection. *For example, $3x + 2y = 5$ and $3x + 2y = 6$ have no solution because the quantity $3x + 2y$ cannot simultaneously be 5 and 6.*
11. Solve and explain word problems leading to two linear equations in two variables.
12. Solve problems involving lines and their equations. *For example, decide whether a point with given coordinates lies on the line with a given equation; construct an equation for a line given two points on the line or one point and the slope; given coordinates for two pairs of points, determine whether the line through the first pair of points intersects the line through the second pair.*

## Functions 8-F

**Function concepts**

1. Understand that a function from one set (called the domain) to another set (called the range) is a rule that assigns to each element of the domain (an input) exactly one element of the range (the corresponding output). The graph of a function is the set of ordered pairs consisting of an input and the corresponding output. *Function notation is not required in Grade 8.*
2. Evaluate expressions that define functions, and solve equations to find the input(s) that correspond to a given output.
3. Compare properties of two functions represented in different ways (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a linear function represented by a table of values and a linear function represented by an algebraic expression, determine which function has the greater rate of change.*

4. Understand that a function is linear if it can be expressed in the form $y = mx + b$ or if its graph is a straight line. *For example, the function $y = x^2$ is not a linear function because its graph contains the points (1,1), (–1,1) and (0,0), which are not on a straight line.*

**Functional relationships between quantities**

5. Understand that functions can describe situations where one quantity determines another.
6. Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship; from two $(x, y)$ values, including reading these from a table; or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.
7. Describe qualitatively the functional relationship between two quantities by reading a graph (e.g., where the function is increasing or decreasing, linear or nonlinear). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.

## Geometry 8-G

**Congruence and similarity**

1. Use coordinate grids to transform figures and to predict the effect of dilations, translations, rotations and reflections.
2. Explain using rigid motions the meaning of congruence for triangles as the equality of all pair of sides and all pairs of angles.
3. Give an informal explanation using rigid motions of the SAS and ASA criteria for triangle congruence, and use them to prove simple theorems.
4. Explain using similarity transformations the meaning of similarity for triangles as the equality of all pairs of angles and the proportionality of all pairs of sides.
5. Give an informal explanation using similarity transformations of the AA and SAS criteria for triangle similarity, and use them to prove simple theorems.

**The Pythagorean Theorem**

6. The side lengths of a right triangle are related by the Pythagorean Theorem. Conversely, if the side lengths of a triangle satisfy the Pythagorean Theorem, it is a right triangle.
7. Explain a proof of the Pythagorean Theorem and its converse.
8. Use the Pythagorean Theorem to determine unknown side lengths in right triangles and to solve problems in two and three dimensions.
9. Use the Pythagorean Theorem to find the distance between two points in a coordinate system.

**Plane and solid geometry**

10. Draw (freehand, with ruler and protractor, and with technology) geometric shapes from given conditions. Focus on constructing triangles from three measures of angles or sides, noticing when the triangle is uniquely defined, ambiguously defined or nonexistent.
11. Understand that slicing a three-dimensional figure with a plane produces a two-dimensional figure. Describe plane sections of right rectangular prisms and right rectangular pyramids.
12. Use hands-on activities to demonstrate and describe properties of: parallel lines in space, the line perpendicular to a given line through a given point, lines perpendicular to a given plane, lines parallel to a given plane, the plane or planes passing through three given points, and the plane perpendicular to a given line at a given point.

## Statistics and Probability 8-SP

**Patterns of association in bivariate data**

1. Understand that scatter plots for bivariate measurement data may reveal patterns of association between two quantities.
2. Construct and interpret scatter plots for bivariate measurement data. Describe patterns such as clustering, outliers, positive or negative association, linear association, nonlinear association.
3. Understand that a straight line is a widely used model for exploring relationships between two quantitative variables. For scatter plots that suggest a linear association, informally fit a straight line, and informally assess the model fit by judging the closeness of the data points to the line.
4. Use the equation of a linear model to solve problems in the context of bivariate measurement data, interpreting the slope and intercept. *For example, in a linear model for a biology experiment, an additional hour of sunlight each day is associated with an additional 1.5 cm in mature plant height.*
5. Understand that patterns of association can also be seen in bivariate categorical data by displaying frequencies and relative frequencies in a two-way table. Construct and interpret a two-way table summarizing data on two categorical variables

collected from the same subjects. Use relative frequencies calculated for rows or columns to describe possible association between the two variables. *For example, collect data from students in your class on whether or not they have a curfew on school nights and whether or not they have assigned chores at home. Is there evidence that those who have a curfew also tend to have chores?*

# Mathematics Standards for High School

## Where is the College-and-Career-Readiness line drawn?

The high school standards specify the mathematics that all students should learn in order to be college and career ready. The high school standards also describe additional mathematics that students should learn to pursue careers and majors in science, technology, engineering and mathematics (STEM) fields. Other forms of advanced work are possible (for example in discrete mathematics or advanced statistics) and can be eventually added to the standards.

Standards beyond the college and career readiness level that are necessary for STEM careers are prefixed with a symbol STEM, as in this example:

> STEM Graph complex numbers in polar form and interpret arithmetic operations on complex numbers geometrically.

Any standard without this tag is understood to be in the common core mathematics curriculum for all students.

## How are the high school standards organized?

The high school standards are listed in conceptual categories, as shown in the Table below. **Appendix A (online) contains drafts of model course descriptions based on these standards**. Conceptual categories portray a coherent view of core high school mathematics; a student's work with Functions, for example, crosses a number of traditional course boundaries, potentially up through and including Calculus.

Conceptual Organization of the High School Standards

| **CCRS** **Draft September 17th** | **High School Standards** **Draft March 10** |
|---|---|
| Number | Number and Quantity |
| Quantity | |
| Expressions | |
| Equations | Algebra |
| Coordinates | |
| Functions | Functions |
| Geometry | Geometry |
| Statistics | Statistics and Probability |
| Probability | |
| Modeling | Modeling** |

* Standards formerly appearing under Coordinates now appear under other headings.

** Making mathematical models is now a Standard for Mathematical Practice. Standards formerly appearing under Modeling are now distributed under other major headings. High school standards with relevance to modeling are flagged with a (★) symbol. A narrative description of modeling remains in the high school standards, but there are no specific standard statements in that narrative description.

# Mathematics | High School—Number and Quantity

**Numbers and Number Systems.** During the years from kindergarten to eighth grade, students must repeatedly extend their conception of number. At first, "number" means "counting number": 1, 2, 3, ... Soon after that, 0 is used to represent "none" and the whole numbers are formed by the counting numbers together with zero. The next extension is fractions. At first, fractions are barely numbers and tied strongly to pictorial representations. Yet by the time students understand division of fractions, they have a strong concept of fractions as numbers and have connected them, via their decimal representations, with the base-ten system used to represent the whole numbers. During middle school, fractions are augmented by negative fractions to form the rational numbers. In Grade 7, students extend this system once more, augmenting the rational numbers with the irrational numbers to form the real numbers. In high school, students will be exposed to yet another extension of number, when the real numbers are augmented by the imaginary numbers to form the complex numbers.

Students sometimes have difficulty accepting new kinds of numbers when these differ in appearance and properties from those of a familiar system. For example, students might decide that complex numbers are not numbers because they are not written with numerical digits, or because they do not describe positive or negative quantities. Indeed, this ascent through number systems makes it fair to ask: what does the word *number* mean that it can mean all of these things? One possible answer is that a number is something that can be used to do mathematics: calculate, solve equations, or represent measurements. Historically, number systems have been extended when there is an intellectual or practical benefit in using the new numbers to solve previously insoluble problems.[1]

Although the referent of "number" changes, the four operations stay the same in important ways. The commutative, associative, and distributive laws extend the properties of operations to the integers, rational numbers, real numbers, and complex numbers. The inverse relationships between addition and subtraction, and multiplication and division are maintained in these larger systems.

Calculators are useful in this strand to generate data for numerical experiments, to help understand the workings of matrix, vector, and complex number algebra, and to experiment with non-integer exponents.

**Quantities.** In their work in measurement up through Grade 8, students primarily measure commonly used attributes such as length, area, volume, and so forth. In high school, students encounter novel situations in which they themselves must conceive the attributes of interest. Such a conceptual process might be called quantification. Quantification is important for science, as when surface area suddenly "stands out" as an important variable in evaporation. Quantification is also important for companies, who must conceptualize relevant attributes and create or choose suitable metrics by which to measure them.

## Content Outline

### The Real Number System

### Quantities

### The Complex Number System

### Vector Quantities and Matrices

[1] See Harel, G., "A Standpoint of Research on Middle/Higher Number and Quantity," a research review provided for the Common Core State Standards Initiative.

## The Real Number System

N-RN

1. Understand that the laws of exponents for positive integer exponents follow from an understanding of exponents as indicating repeated multiplication, and from the associative law for multiplication.
2. Understand that the definition of the meaning of zero, positive rational, and negative exponents follows from extending the laws of exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, since* $(5^{1/3})^3 = 5^{(1/3)\cdot 3} = 5^1 = 5$, $5^{1/3}$ *is a cube root of* $5$.
3. Understand that sums and products of rational numbers are rational.
4. Understand that the sum of a rational number and an irrational number is irrational, and that the product of a nonzero rational number and an irrational number is irrational.
5. Rewrite expressions using the laws of exponents. *For example,* $(5^{1/2})^3 = 5^{3/2}$ *and* $1/5 = 5^{-1}$.

## Quantities★

N-Q

1. Understand that the magnitude of a quantity is independent of the unit used to measure it. *For example, the density of a liquid does not change when it is measured in another unit. Rather, its measure changes. The chosen unit "measures" the quantity by giving it a numerical value ("the density of lead is 11.3 times that of water").*
2. Use units as a way to understand problems and to guide the solution of multi-step problems, involving, e.g., acceleration, currency conversions, derived quantities such as person-hours and heating degree days, social science rates such as per-capita income, and rates in everyday life such as points scored per game.
3. Define metrics for the purpose of descriptive modeling. *For example, find a good measure of overall highway safety; propose and debate measures such as fatalities per year, fatalities per year per driver, or fatalities per vehicle-mile traveled.*
4. Add, subtract, multiply, and divide numbers expressed in scientific notation, including problems where both decimal and scientific notation are used. Use scientific notation and choose units of appropriate size for measurements of very large or very small quantities (e.g., use millimeters per year for seafloor spreading). Interpret scientific notation that has been generated by technology.
5. Use and interpret quantities and units correctly in algebraic formulas.
6. Use and interpret quantities and units correctly in graphs and data displays (function graphs, data tables, scatter plots, and other visual displays of quantitative information). Generate graphs and data displays using technology.

## The Complex Number System

N-CN

1. Understand that the relation $i^2 = -1$ and the commutative, associative, and distributive laws can be used to calculate with complex numbers.
2. STEM Understand that polynomials can be factored over the complex numbers, e.g., as in $x^2 + 4 = (x + 2i)(x - 2i)$.
3. STEM Understand that complex numbers can be visualized on the complex plane. Real numbers correspond to points on the horizontal (real) axis, and imaginary numbers to points on the vertical axis.
4. STEM Understand that on the complex plane, arithmetic of complex numbers can be interpreted geometrically: addition is analogous to vector addition, and multiplication can be understood as rotation and dilation about the origin. Complex conjugation is reflection across the real axis.
5. STEM Understand that on the complex plane, as on the real line, the distance between numbers is the absolute value of the difference, and the midpoint of a segment is the average of the numbers at its endpoints.
6. Add, subtract, and multiply complex numbers.
7. STEM Find the conjugate of a complex number; use conjugates to find absolute values and quotients of complex numbers.
8. STEM Solve quadratic equations with real coefficients that have complex solutions using a variety of methods.
9. STEM Graph complex numbers in rectangular form.
10. STEM Graph complex numbers in polar form and interpret arithmetic operations on complex numbers geometrically.
11. STEM Explain why the rectangular and polar forms of a complex number represent the same number.

★ Standard with close connection to modeling.

1. STEM Understand that vector quantities have both magnitude and direction. Vector quantities are typically represented by directed line segments. The magnitude of a vector **v** is commonly denoted $|\mathbf{v}|$ or $||\mathbf{v}||$.
2. STEM Understand that vectors are determined by the coordinates of their initial and terminal points, or by their components.
3. STEM Understand that vectors can be added end-to-end, component-wise, or by the parallelogram rule. The magnitude of a sum of two vectors is typically not the sum of the magnitudes.
4. STEM Understand that a vector **v** can be multiplied by a real number $c$ (called a scalar in this context) to form a new vector $c\mathbf{v}$ with magnitude $|c|v$. When $|c|v \neq 0$, the direction of $c\mathbf{v}$ is either along **v** (for $c > 0$) or against **v** (for $c < 0$). Scalar multiplication can be shown graphically by scaling vectors and possibly reflecting them in the origin; scalar multiplication can also be performed component-wise, e.g., as $c(v_x, v_y) = (cv_x, cv_y)$.
5. STEM Understand that vector subtraction $\mathbf{v} - \mathbf{w}$ is defined as $\mathbf{v} + (-\mathbf{w})$. Two vectors can be subtracted graphically by connecting the tips in the appropriate order.
6. STEM Understand that matrices can be multiplied by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled. Matrices of the same dimensions can be added or subtracted. Matrices with compatible dimensions can be multiplied. Unlike multiplication of numbers, matrix multiplication is not a commutative operation, but still satisfies the associative and distributive laws.
7. STEM Understand that a vector, when regarded as a matrix with one column, can be multiplied by a matrix of suitable dimensions to produce another vector. A $2 \times 2$ matrix can be viewed as a transformation of the plane.
8. STEM Understand that a system of linear equations can be represented as a single matrix equation in a vector variable.
9. STEM Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.
10. STEM Perform basic vector operations (addition, subtraction, scalar multiplication) both graphically and algebraically.
11. STEM Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.
12. STEM Solve problems involving velocity and quantities that can be represented by vectors.★
13. STEM Add, subtract, and multiply matrices of appropriate dimensions.
14. STEM Use matrices to store and manipulate data, e.g., to represent payoffs or incidence relationships in a network.
15. STEM Represent systems of linear equations as matrix equations.
16. STEM Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension greater than $3 \times 3$).

★ Standard with close connection to modeling.

# Mathematics | High School—Algebra

**Expressions.** An expression is a description of a computation on numbers and symbols that represent numbers, using arithmetic operations and the operation of raising a number to rational exponents. Conventions about the use of parentheses and the order of operations assure that each expression is unambiguous. Creating an expression that describes a computation involving a general quantity requires the ability to express the computation in general terms, abstracting from specific instances.

Reading an expression with comprehension involves analysis of its underlying structure. This may suggest a different but equivalent way of writing the expression that exhibits some different aspect of its meaning. For example, $p + 0.05p$ can be interpreted as the addition of a 5% tax to a price $p$. Rewriting $p + 0.05p$ as $1.05p$ shows that adding a tax is the same as multiplying the price by a constant factor.

Algebraic manipulations are governed by deductions from the commutative, associative, and distributive laws and the inverse relationships between the four operations, and the conventions of algebraic notation. These extend what students have learned about arithmetic expressions in K–8 to expressions that involve exponents, radicals, and representations of real numbers, and, for STEM-intending students, complex numbers.

At times, an expression is the result of applying operations to simpler expressions. Viewing such an expression by singling out these simpler expressions can sometimes clarify its underlying structure.

A spreadsheet or a CAS environment can be used to experiment with algebraic expressions, perform complex algebraic manipulations,and understand how algebraic manipulations behave.

**Equations and inequalities.** An equation is a statement that two expressions are equal. Solutions to an equation are numbers that make the equation true when assigned to the variables in it. If the equation is true for all numbers, then it is called an identity; identities are often discovered by using the laws of arithmetic or the laws of exponents to transform one expression into another.

The solutions of an equation in one variable form a set of numbers; the solutions of an equation in two variables form a set of ordered pairs of numbers, which can be graphed in the coordinate plane. Two or more equations and/or inequalities form a system. A solution for such a system must satisfy every equation and inequality in the system.

An equation can often be solved by successively transforming it into one or more simpler equations. The process is governed by deductions based on the properties of equality. For example, one can add the same constant to both sides without changing the solutions, but squaring both sides might lead to extraneous solutions. Strategic competence in solving includes looking ahead for productive manipulations and anticipating the nature and number of solutions.

Some equations have no solutions in a given number system, stimulating the extension of that system. For example, the solution of $x + 1 = 0$ is an integer, not a whole number; the solution of $2x + 1 = 0$ is a rational number, not an integer; the solutions of $x^2 - 2 = 0$ are real numbers, not rational numbers; and the solutions of $x^2 + 2 = 0$ are complex numbers, not real numbers.

The same solution techniques used to solve equations can be used to rearrange formulas. For example, the formula for the area of a trapezoid, $A = ((b_1+b_2)/2)h$, can be solved for $h$ using the same deductive process.

Inequalities can be solved by reasoning about the properties of inequality. Many, but not all, of the properties of equality continue to hold for inequalities and can be useful in solving them.

*Connections to Functions and Modeling*. Expressions can define functions, and equivalent expressions define the same function. Equations in two variables may also define functions. Asking when two functions have the same value leads to an equation; graphing the two functions allows for the approximate solution of the equation. Converting a verbal description to an equation, inequality, or system of these is an essential skill in modeling.

## Content Outline

**Seeing Structure in Expressions**

**Arithmetic with Polynomials and Rational Expressions**

**Creating Equations that Describe Numbers or Relationships**

**Reasoning with Equations and Inequalities**

## Seeing Structure in Expressions A-SSE

1. Understand that different forms of an expression may reveal different properties of the quantity in question; a purpose in transforming expressions is to find those properties. *Examples: factoring a quadratic expression reveals the zeros of the function it defines, and putting the expression in vertex form reveals its maximum or minimum value; the expression* $1.15^t$ *can be rewritten as* $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*
2. Understand that complicated expressions can be interpreted by viewing one or more of their parts as single entities.
3. Interpret an expression that represents a quantity in terms of the context. *Include interpreting parts of an expression, such as terms, factors and coefficients.* ★
4. Factor, expand, and complete the square in quadratic expressions.
5. See expressions in different ways that suggest ways of transforming them. *For example, see* $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, *thus recognizing it as a difference of squares that can be factored as* $(x^2 - y^2)(x^2 + y^2)$.
6. Rewrite expressions using the laws of exponents. *For example,* $(x^{1/2})^3 = x^{3/2}$ *and* $1/x = x^{-1}$.
7. Use the laws of exponents to interpret expressions for exponential functions, recognizing positive rational exponents as indicating roots of the base and negative exponents as indicating the reciprocal of a power. *For example, identify the per unit percentage change in functions such as* $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, $y = (1.2)^{t/10}$, *and conclude whether it represents exponential growth or decay. Recognize that any nonzero number raised to the zero power is 1, for example,* $12(1.05)^0 = 12$. *Avoid common errors such as confusing* $6(1.05)^t$ *with* $(6\cdot 1.05)^t$ *and* $5(0.03)^t$ *with* $5(1.03)^t$.
8. STEM Prove the formula for the sum of a geometric series, and use the formula to solve problems.

## Arithmetic with Polynomials and Rational Expressions A-APR

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication.
2. Understand that polynomial identities become true statements no matter which real numbers are substituted. *For example, the polynomial identity* $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ *can be used to generate Pythagorean triples.*
3. Understand the Remainder Theorem: For a polynomial $p(x)$ and a number $a$, the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.
4. STEM Understand that the Binomial Theorem gives the expansion of $(x + a)^n$ in powers of $x$ for a positive integer $n$ and a real number $a$, with coefficients determined for example by Pascal's Triangle. The Binomial Theorem can be proved by mathematical induction or by a combinatorial argument.
5. STEM Understand that rational expressions are quotients of polynomials. They form a system analogous to the rational numbers, closed under division by a nonzero rational function.
6. Add, subtract and multiply polynomials.
7. Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the polynomial.
8. Transform simple rational expressions using the commutative, associative, and distributive laws, and the inverse relationship between multiplication and division.
9. Divide a polynomial $p(x)$ by a divisor of the form $x - a$ using long division.
10. STEM Identify zeros and asymptotes of rational functions, when suitable factorizations are available, and use the zeros and asymptotes to construct a rough graph of the function.
11. STEM Divide polynomials, using long division for linear divisors and long division or a computer algebra system for higher degree divisors.

## Creating Equations That Describe Numbers or Relationships A-CED★

1. Understand that equations in one variable are often created to describe properties of a specific but unknown number.
2. Understand that equations in two or more variables that represent a relationship between quantities can be built by experimenting with specific numbers in the relationship.
3. Write equations and inequalities that specify an unknown quantity or to express a relationship between two or more quantities. Use the equations and inequalities to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*

★ Standard with close connection to modeling.

4. Rearrange formulas to highlight a quantity of interest. *For example, transform Ohm's law $V = IR$ to highlight resistance $R$; in motion with constant acceleration, transform $v_{f,x}^2 - v_{i,x}^2 = 2a_x(x_f - x_i)$ to highlight the change in position along the x-axis, $x_f - x_i$.*

## Reasoning with Equations and Inequalities

A-REI

1. Understand that to solve an equation algebraically, one makes logical deductions from the equality asserted by the equation, often in steps that replace it with a simpler equation whose solutions include the solutions of the original one.
2. Understand that the method of completing the square can transform any quadratic equation in $x$ into an equivalent equation of the form $(x - p)^2 = q$. This leads to the quadratic formula.
3. Understand that given a system of two linear equations in two variables, adding a multiple of one equation to another produces a system with the same solutions. This principle, combined with principles already encountered with equations in one variable, allows for the simplification of systems.
4. Understand that the graph of an equation in two variables is the set of its solutions plotted in the coordinate plane, often forming a curve or a line.
5. Understand that solutions to two equations in two variables correspond to points of intersection of their graphs, because points of intersection satisfy both equations simultaneously.
6. Understand that the solutions to a linear inequality in two variables can be graphed as a half-plane (excluding the boundary in the case of a strict inequality).
7. Understand that solutions to several linear inequalities in two variables correspond to points in the intersection of the regions in the plane defined by the solutions to the inequalities.
8. Understand that equations and inequalities can be viewed as constraints in a problem situation, e.g., inequalities describing nutritional and cost constraints on combinations of different foods. ★
9. STEM Understand that the relationship between an invertible function $f$ and its inverse function can be used to solve equations of the form $f(x) = c$.
10. Solve simple rational and radical equations in one variable, noting and explaining extraneous solutions.
11. Solve linear equations in one variable, including equations with coefficients represented by letters.
12. Solve quadratic equations in one variable. *Include methods such as inspection (e.g. for $x^2 = 49$), square roots, completing the square, the quadratic formula and factoring. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers $a$ and $b$.*
13. Solve equations $f(x) = g(x)$ approximately by finding the intersections of the graphs of $f(x)$ and $g(x)$, e.g. using technology to graph the functions. *Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, exponential, and logarithmic functions.*
14. Solve linear inequalities in one variable and graph the solution set on a number line.
15. Solve systems of linear equations algebraically and graphically, focusing on pairs of linear equations in two variables.
16. Solve algebraically a simple system consisting of one linear equation and one quadratic equation in two variables; for example, find points of intersection between the line $y = -3x$ and the circle $x^2 + y^2 = 3$.
17. Graph the solution set of a system of linear inequalities in two variables.
18. In modeling situations, represent constraints by systems of equations and/or inequalities, and interpret solutions of these systems as viable or non-viable options in the modeling context. ★
19. In the context of exponential models, solve equations of the form $a\,b^{ct} = d$ where $a$, $c$, and $d$ are specific numbers and the base $b$ is 2, 10, or $e$.★
20. STEM Relate the properties of logarithms to the laws of exponents and solve equations involving exponential functions.
21. STEM Use inverse functions to solve equations of the form $a\sin(bx + c) = d$, $a\cos(bx + c) = d$, and $a\tan(bx + c) = d$.

★ Standard with close connection to modeling.

# Mathematics | High School—Functions

Functions describe situations where one quantity determines another. For example, the return on $10,000 invested at an annualized percentage rate of 4.25% is a function of the length of time the money is invested. Because nature and society are full of dependencies between quantities, functions are important tools in the construction of mathematical models.

In school mathematics, functions usually have numerical inputs and outputs and are often defined by an algebraic expression. For example, the time in hours it takes for a car to drive 100 miles is a function of the car's speed in miles per hour, $v$; the rule $T(v) = 100/v$ expresses this relationship algebraically and defines a function whose name is $T$.

The set of inputs to a function is called its domain. We often infer the domain to be all inputs for which the expression defining a function has a value, or for which the function makes sense in a given context.

A function can be described in various ways, such as by a graph (e.g., the trace of a seismograph); by a verbal rule, as in, "I'll give you a state, you give me the capital city"; or by an algebraic expression like $f(x) = a + bx$. The graph of a function is often a useful way of visualizing the relationship the function models, and manipulating a mathematical expression for a function can throw light on the function's properties. Graphing technology and spreadsheets are also useful tools in the study of functions.

Functions presented as expressions can model many important phenomena. Two important families of functions characterized by laws of growth are linear functions, which grow at a constant rate, and exponential functions, which grow at a constant percent rate. Linear functions with a constant term of zero describe proportional relationships.

A graphing utility or a CAS can be used to experiment with properties of the functions and their graphs and to build computational models of functions, including recursively defined functions.

*Connections to Expressions, Equations, Modeling and Coordinates*. Determining an output value for a particular input involves evaluating an expression; finding inputs that yield a given output involves solving an equation. Questions about when two functions have the same value lead to equations, whose solutions can be visualized from the intersection of their graphs. Because functions describe relationships between quantities, they are frequently used in modeling. Sometimes functions are defined by a recursive process, which can be displayed effectively using a spreadsheet or other technology.

## Content Outline

**Interpreting Functions**

**Building Functions**

**Linear, Quadratic, and Exponential Models**

**Trigonometric Functions**

Limits and Continuity†

Differential Calculus†

Applications of Derivatives†

Integral Calculus†

Applications of Integration†

Infinite Series†

† Specific standards for calculus domains are not listed.

## Interpreting Functions A-IF

1. Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If $f$ is a function and $x$ is an element of its domain, then $f(x)$ denotes the output of $f$ corresponding to the input $x$.
2. Understand that functions of a single variable have key characteristics, including: zeros; extreme values; average rates of change (over intervals); intervals of increasing, decreasing and/or constant behavior; and end behavior.
3. Understand that a function defined by an expression may be written in different but equivalent forms, which can reveal different properties of the function.
4. Use function notation and evaluate functions for inputs in their domains.
5. Describe qualitatively the functional relationship between two quantities by reading a graph (e.g., where the function is increasing or decreasing, what its long-run behavior appears to be, and whether it appears to be periodic).★
6. Sketch a graph that exhibits the qualitative features of a function that models a relationship between two quantities.★
7. Compare properties of two functions represented in different ways (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, draw conclusions about the graph of a quadratic function from its algebraic expression.*
8. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★
9. Describe the qualitative behavior of functions presented in graphs and tables. *Identify: intercepts; intervals where the function is increasing, decreasing, positive or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★
10. Use technology to exhibit the effects of parameter changes on the graphs of linear, power, quadratic, square root, cube root, and polynomial functions, and simple rational, exponential, logarithmic, sine, cosine, absolute value, and step functions.★
11. Transform quadratic polynomials algebraically to reveal different features of the function they define, such as zeros, extreme values, and symmetry of the graph.

## Building Functions A-BF

1. Understand that functions can be described by specifying an explicit expression, a recursive process or steps for calculation.
2. Understand that sequences are functions whose domain is a subset of the nonnegative integers.
3. STEM Understand that composing a function $f$ with a function $g$ creates a new function called the composite function—for an input number $x$, the output of the composite function is $f(g(x))$.
4. STEM Understand that the inverse of an invertible function "undoes" what the function does; that is, composing the function with its inverse in either order returns the original input. One can sometimes produce an invertible function from a non-invertible function by restricting the domain (e.g., squaring is not an invertible function on the real numbers, but squaring is invertible on the nonnegative real numbers).
5. Write a function that describes a relationship between two quantities, for example by varying parameters in and combining standard function types (such as linear, quadratic or exponential functions). Use technology to experiment with parameters and to illustrate an explanation of the behavior of the function when parameters vary.★
6. Solve problems involving linear, quadratic, and exponential functions.★
7. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of $k$ (both positive and negative); find the value of $k$ given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology.
8. Generate an arithmetic or geometric sequence given a recursive rule for the sequence.★
9. As a way to describe routine modeling situations, write arithmetic and geometric sequences both recursively and in closed form, and translate between the two forms.★
10. STEM Evaluate composite functions and compose functions symbolically.
11. STEM Read values of an inverse function from a graph or a table, given that the function has an inverse.
12. STEM For linear or simple exponential functions, find a formula for an inverse function by solving an equation.
13. STEM Verify symbolically by composition that one function is the inverse of another.

## Linear, Quadratic, and Exponential Models A-LQE

1. Understand that a linear function, defined by $f(x) = mx + b$ for some constants $m$ and $b$, models a situation in which a quantity changes at a constant rate, $m$, relative to another.★
2. Understand that quadratic functions have maximum or minimum values and can be used to model problems with optimum solutions.★
3. Understand that an exponential function, defined by $f(x) = ab^x$ or by $f(x) = a(1 + r)^x$ for some constants $a, b > 0$ and $r > -1$, models a situation where a quantity grows or decays by a constant factor or a constant percentage change over each unit interval.★
4. Understand that linear functions grow by equal differences over equal intervals; exponential functions grow by equal factors over equal intervals.★
5. Understand that in an arithmetic sequence, differences between consecutive terms form a constant sequence, and second differences are zero. Conversely, if the second differences are zero, the sequence is arithmetic. Arithmetic sequences can be seen as linear functions.★
6. Understand that in a sequence that increases quadratically (e.g., $a_n = 3n^2 + 2n + 1$), differences between consecutive terms form an arithmetic sequence, and second differences form a constant sequence. Conversely, if the second differences form a constant sequence with nonzero value, the sequence increases quadratically.★
7. Understand that in a geometric sequence, ratios of consecutive terms are all the same.★
8. Understand that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.★
9. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★
10. Construct a function to describe a linear relationship between two quantities. Determine the rate of change and constant term of a linear function from a graph, a description of a relationship, or from two $(x, y)$ values (include reading these from a table).★
11. Use quadratic functions to model problems, e.g., in situations with optimum solutions.★
12. Construct an exponential function in the form $f(x) = a(1 + r)^x$ or $f(x) = ab^x$ to describe a relationship in which one quantity grows with respect to another at a constant percent growth rate or a with a constant growth factor.★
13. Interpret the rate of change and constant term of a linear function or sequence in terms of the situation it models, and in terms of its graph or a table of values.★
14. Calculate and interpret the growth factor for an exponential function (presented symbolically or as a table) given a fixed interval. Estimate the growth factor from a graph.★
15. Recognize a quantitative relationship as linear, exponential, or neither from description of a situation.★
16. Compare quantities increasing exponentially to quantities increasing linearly or as a polynomial function.★

## Trigonometric Functions F-TF

1. STEM Understand that the unit circle in the coordinate plane enables one to define the sine, cosine, and tangent functions for real numbers.
2. STEM Understand that trigonometric functions are periodic by definition, and sums and products of functions with the same period are periodic.
3. STEM Understand that restricting trigonometric functions to a domain on which they are always increasing or always decreasing allows for the construction of an inverse function.
4. STEM Revisit trigonometric functions and their graphs in terms of radians.
5. STEM Use the unit circle to determine geometrically the values of sine, cosine, tangent for integer multiples of $\pi/4$ and $\pi/6$.
6. STEM Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.
7. STEM Solve simple trigonometric equations formally using inverse trigonometric functions and evaluate the solutions numerically using technology. *Solving trigonometric equations by means of the quadratic formula is optional.*

## Limits and Continuity† F-LC

★ Standard with close connection to modeling.

† Specific standards for calculus domains are not listed.

## Differential Calculus[†] F-DC

## Applications of Derivatives[†] F-AD

## Integral Calculus[†] F-IC

## Applications of Integration[†] F-AI

## Infinite Series[†] F-IS

[†] Specific standards for calculus domains are not listed.

# Mathematics | High School—Modeling

Modeling links classroom mathematics and statistics to everyday life, work, and decision-making. Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions. Quantities and their relationships in physical, economic, public policy, social and everyday situations can be modeled using mathematical and statistical methods. When making mathematical models, technology is valuable for varying assumptions, exploring consequences, and comparing predictions with data.

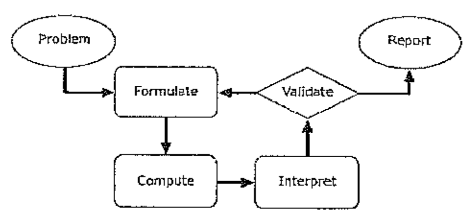
A model can be very simple, such as writing total cost as a product of unit price and number bought, or using a geometric shape to describe a physical object like a coin. Even such simple models involve making choices. It is up to us whether to model a coin as a three-dimensional cylinder, or whether a two-dimensional disk works well enough for our purposes. Other situations—modeling a delivery route, a production schedule, or a comparison of loan amortizations—need more elaborate models that use other tools from the mathematical sciences. Real-world situations are not organized and labeled for analysis; formulating tractable models, representing such models, and analyzing them is appropriately a creative process. Like every such process, this depends on acquired expertise as well as creativity.

Some examples of such situations might include:

- Estimating how much water and food is needed for emergency relief in a devastated city of 3 million people, and how it might be distributed.
- Planning a table tennis tournament for 7 players at a club with 4 tables, where each player plays against each other player.
- Designing the layout of the stalls in a school fair so as to raise as much money as possible.
- Analyzing stopping distance for a car.
- Modeling savings account balance, bacterial colony growth, or investment growth.
- Critical path analysis, e.g., applied to turnaround of an aircraft at an airport.
- Risk situations, like extreme sports, pandemics and terrorism.
- Relating population statistics to individual predictions.

In situations like these, the models devised depend on a number of factors: How precise an answer do we want or need? What aspects of the situation do we most need to understand, control, or optimize? What resources of time and tools do we have? The range of models that we can create and analyze is also constrained by the limitations of our mathematical, statistical, and technical skills, and our ability to recognize significant variables and relationships among them. Diagrams of various kinds, spreadsheets and other technology, and algebra are powerful tools for understanding and solving problems drawn from different types of real-world situations.

One of the insights provided by mathematical modeling is that essentially the same mathematical or statistical structure can model seemingly different situations. Models can also shed light on the mathematical structures themselves, for example as when a model of bacterial growth makes more vivid the explosive growth of the exponential function.




The basic modeling cycle is summarized in the diagram. It involves (1) identifying variables in the situation and selecting those than represent essential features, (2) formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables, (3) analyzing and performing operations on these relationships to draw conclusions, (4) interpreting the results of the mathematics in terms of the original situation, (5) validating the conclusions by comparing them with the situation, and then, either improving the model or, if it is acceptable, (6) reporting on the conclusions and the reasoning behind them. Choices, assumptions and approximations are present throughout this cycle.

In descriptive modeling, a model simply describes the phenomena or summarizes them in a compact form. Graphs of observations are a familiar descriptive model—for example, graphs of global temperature and atmospheric $CO_2$ over time.

Analytic modeling seeks to explain data on the basis of deeper theoretical ideas, albeit with parameters that are empirically based; for example, exponential growth of bacterial colonies (until cut-off mechanisms such as pollution or starvation intervene) follows from a constant reproduction rate. Functions are an important tool for analyzing such problems.

Graphing utilities, spreadsheets, CAS environments, and dynamic geometry software are powerful tools that can be used to model purely mathematical phenomena (e.g., the behavior of polynomials) as well as physical phenomena.

## Modeling Standards

*Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (★).*

# Mathematics | High School—Statistics and Probability★

Decisions or predictions are often based on data—numbers in context. These decisions or predictions would be easy if the data always sent a clear message, but the message is often obscured by variability. Statistics provides tools for describing variability in data and for making informed decisions that take it into account.

Data are gathered, displayed, summarized, examined, and interpreted to discover patterns and deviations from patterns. Quantitative data can be described in terms of key characteristics: measures of shape, center, and spread. The shape of a data distribution might be described as symmetric, skewed, flat, or bell shaped, and it might be summarized by a statistic measuring center (such as mean or median) and a statistic measuring spread (such as standard deviation or interquartile range). Different distributions can be compared numerically using these statistics or compared visually using plots. Knowledge of center and spread are not enough to describe a distribution. Which statistics to compare, which plots to use, and what the results of a comparison might mean, depend on the question to be investigated and the real-life actions to be taken.

Randomization has two important uses in drawing statistical conclusions. First, collecting data from a random sample of a population makes it possible to draw valid conclusions about the whole population, taking variability into account. Second, randomly assigning individuals to different treatments allows a fair comparison of the effectiveness of those treatments. A statistically significant outcome is one that is unlikely to be due to chance alone, and this can be evaluated only under the condition of randomness. The conditions under which data are collected are important in drawing conclusions from the data; in critically reviewing uses of statistics in public media and other reports it is important to consider the study design, how the data were gathered, and the analyses employed as well as the data summaries and the conclusions drawn.

Random processes can be described mathematically by using a probability model. One begins to make a probability model by listing or describing the possible outcomes (the sample space) and assigning probabilities. In situations such as flipping a coin, rolling a number cube, or drawing a card, it might be reasonable to assume various outcomes are equally likely. In a probability model, sample points represent outcomes and combine to make up events; probabilities of events can be computed by applying the additive and multiplicative laws of probability. Interpreting these probabilities relies on an understanding of independence and conditional probability, which can be approached through the analysis of two-way tables.

Technology plays an important role in statistics and probability by making it possible to generate plots, functional models, and correlation coefficients, and to simulate many possible outcomes in a short amount of time.

*Connections to Functions and Modeling.* Functional models may be used to approximate data; if the data are approximately linear, the relationship may be modeled with a regression line and the strength and direction of such a relationship may be expressed through a correlation coefficient.

## Content Outline

### Summarizing Categorical and Measurement Data

### Probability Models

### Independently Combined Probability Models

### Making Inferences and Justifying Conclusions Drawn from Data

### Conditional Probability and the Laws of Probability

### Experimenting and Simulating to Model Probabilities

### Using Probability to Make Decisions

★ Most or all of the standards in Statistics and Probability have a close connection to modeling.

## Summarizing Categorical and Quantitative Data S-SI

1. Understand that statistical methods take variability into account to support making informed decisions based on data collected to answer specific questions.
2. Understand that visual displays and summary statistics condense the information in data sets into usable knowledge.
3. Understand that patterns of association or relationships between variables may emerge through careful analysis of multi-variable data.
4. Summarize comparative or bivariate categorical data in two-way frequency tables. Interpret joint, marginal and conditional relative frequencies in the context of the data, recognizing possible associations and trends in bivariate categorical data.
5. Compare data on two or more count or measurement variables by using plots on the real number line (dot plots, histograms, and box plots). Use statistics appropriate to the shape of the data distribution to summarize center (median, mean) and spread (interquartile range, standard deviation) of the data sets. Interpret changes in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers).
6. Represent bivariate quantitative data on a scatter plot and describe how the variables are related.
7. Fit a linear function for scatter plots that suggest a linear association. Informally assess the fit of the model function by plotting and analyzing residuals.
8. Use a model function fitted to the data to solve problems in the context of the data, interpreting the slope (rate of change) and the intercept (constant term).
9. Compute (using technology) and interpret the correlation coefficient for a linear relationship between variables.
10. Distinguish between correlation and causation.

## Probability Models S-PM

1. Understand that in a probability model, individual outcomes have probabilities that sum to 1. When outcomes are categorized, the probability of a given type of outcome is the sum of the probabilities of all the individual outcomes of that type.
2. Understand that uniform probability models are useful models for processes such as (i) the selection of a person from a population; (ii) the selection of a number in a lottery; (iii) any physical situation in which symmetry suggests that different individual outcomes are equally likely.
3. Understand that two different empirical probability models for the same process will rarely assign exactly the same probability to a given type of outcome. But if the data sets are large and the methods used to collect the data for the two data sets are consistent, the agreement between the models is likely to be reasonably good.
4. Understand that a (theoretical) uniform probability model may be judged by comparing it to an empirical probability model for the same process. If the theoretical assumptions are appropriate and the data set is large, then the two models should agree approximately. If the agreement is not good, then it may be necessary to modify the assumptions underlying the theoretical model or look for factors that might have affected the data used to create the empirical model.
5. Use a uniform probability model to compute probabilities for a process involving uncertainty, including the random selection of a person from a population and physical situations where symmetry suggests that different individual outcomes are equally likely.
   a. List the individual outcomes to create a sample space.
   b. Label the individual outcomes in the sample space to reflect important characteristics or quantities associated with them.
   c. Determine probabilities of individual outcomes, and determine the probability of a type or category of outcome as the fraction of individual outcomes it includes.
6. Generate data by sampling, repeated experimental trials, and simulations. Record and appropriately label such data, and use them to construct an empirical probability model. Compute probabilities in such models.
7. Compare probabilities from a theoretical model to probabilities from a corresponding empirical model for the same situation. If the agreement is not good, explain possible sources of the discrepancies.

## Independently Combined Probability Models S-IPM

1. Understand that to describe a pair of random processes (such as tossing a coin and rolling a number cube), or one random process repeated twice (such as randomly selecting a student in the class on two different days), two probability models can be combined into a single model.

a. The sample space for the combined model is formed by listing all possible ordered pairs that combine an individual outcome from the first model with an individual outcome from the second. Each ordered pair is an individual outcome in the combined model.

b. The total number of individual outcomes (ordered pairs) in the combined model is the product of the number of individual outcomes in each of the two original models.

2. Understand that when two probability models are combined independently, the probability that one type of outcome in the first model occurs together with another type of outcome in the second model is the product of the two corresponding probabilities in the original models (the Multiplication Rule).

3. Combine two uniform models independently to compute probabilities for a pair of random processes (e.g., flipping a coin twice, selecting one person from each of two classes).

   a. Use organized lists, tables and tree diagrams to represent the combined sample space.

   b. Determine probabilities of ordered pairs in the combined model, and determine the probability of a particular type or category of outcomes in the combined model, as the fraction of ordered pairs corresponding to it.

4. For two independently combined uniform models, use the Multiplication Rule to determine probabilities.

## Making Inferences and Justifying Conclusions S-IC

1. Understand that statistics is a process for making inferences about population parameters based on a sample from that population; randomness is the foundation for statistical inference.
2. Understand that the design of an experiment or sample survey is of critical importance to analyzing the data and drawing conclusions.
3. Understand that simulation-based techniques are powerful tools for making inferences and justifying conclusions from data.
4. Use probabilistic reasoning to decide if a specified model is consistent with results from a given data-generating process. (For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?)
5. Recognize the purposes of and differences among sample surveys, experiments and observational studies; explain how randomization relates to each.
6. Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.
7. Use data from a randomized experiment to compare two treatments; justify significant differences between parameters through the use of simulation models for random assignment.
8. Evaluate reports based on data.

## Conditional Probability and the Laws of Probability S-CP

1. Understand that events are subsets of a sample space; often, events of interest are defined by using characteristics (or categories) of the sample points, or as unions, intersections, or complements thereof ("and," "or," "not"). A sample point may belong to several events (categories).
2. Understand that if A and B are two events, then in a uniform model the conditional probability of A given B, denoted by $P(A \mid B)$, is the fraction of B's sample points that also lie in A.
3. Understand that the laws of probability allow one to use known probabilities to determine other probabilities of interest.
4. Compute probabilities by constructing and analyzing sample spaces, representing them by tree diagrams, systematic lists, and Venn diagrams.
5. Use the laws of probability to compute probabilities.
6. Apply concepts such as intersections, unions and complements of events, and conditional probability and independence to define or analyze events, calculate probabilities and solve problems.
7. Construct and interpret two-way tables to show probabilities when two characteristics (or categories) are associated with each sample point. Use a two-way table to determine conditional probabilities.★
8. Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations.★
9. Use permutations and combinations to compute probabilities of compound events and solve problems.

★ Standard with close connection to modeling.

## Experimenting and Simulating to Model Probabilities S-ES

1. Understand that sets of data obtained from surveys, simulations or other means can be used as probability models, by treating the data set itself as a sample space, in which the sample points are the individual pieces of data.
2. Understand that the probability of an outcome can be interpreted as an assertion about the long-run proportion of the outcome's occurrence if the random experiment is repeated a large number of times.
3. Calculate experimental probabilities by performing simulations or experiments involving a probability model and using relative frequencies of outcomes.
4. Compare the results of simulations with predicted probabilities. When there are substantial discrepancies between predicted and observed probabilities, explain them.
5. Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets and tables to estimate areas under the normal curve.

## Using Probability to Make Decisions S-MD

1. Understand that the expected value of a random variable is the weighted average of its possible values, with weights given by their respective probabilities.
2. Understand that when the possible outcomes of a decision can be assigned probabilities and payoff values, the decision can be analyzed as a random variable with an expected value, e.g., of an investment.
3. Calculate expected value, e.g. to determine the fair price of an investment.
4. Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).
5. Evaluate and compare two investments or strategies with the same expected value, where one investment or strategy is safer than the other.
6. Evaluate and compare two investments or strategies, where one investment or strategy is safer but has lower expected value. Include large and small investments, and situations with serious consequences.
7. Analyze decisions and strategies using probability concepts (e.g. product testing, medical testing, pulling a hockey goalie at the end of a game).

# Mathematics | High School—Geometry

An understanding of the attributes and relationships of geometric objects can be applied in diverse contexts—interpreting a schematic drawing, estimating the amount of wood needed to frame a sloping roof, rendering computer graphics, or designing a sewing pattern for the most efficient use of material.

Understanding the attributes of geometric objects often relies on measurement: a circle is a set of points in a plane at a fixed distance from a point; a cube is bounded by six squares of equal area; when two parallel lines are crossed by a transversal, pairs of corresponding angles are congruent.

The concepts of congruence, similarity and symmetry can be united under the concept of geometric transformation. Reflections and rotations each explain a particular type of symmetry, and the symmetries of an object offer insight into its attributes—as when the reflective symmetry of an isosceles triangle assures that its base angles are congruent. Applying a scale transformation to a geometric figure yields a similar figure. The transformation preserves angle measure, and lengths are related by a constant of proportionality.

The definitions of sine, cosine and tangent for acute angles are founded on right triangle similarity, and, with the Pythagorean theorem, are fundamental in many real-world and theoretical situations.

Coordinate geometry is a rich field for exploration. How does a geometric transformation such as a translation or reflection affect the coordinates of points? How is the geometric definition of a circle reflected in its equation? Coordinates can describe locations in three dimensions and extend the use of algebraic techniques to problems involving the three-dimensional world we live in.

Dynamic geometry environments provide students with experimental and modeling tools that allow them to investigate geometric phenomena in much the same was as CAS environments allow them to experiment with algebraic phenomena.

*Connections to Equations and Inequalities.* The correspondence between numerical coordinates and geometric points allows methods from algebra to be applied to geometry and vice versa. The solution set of an equation becomes a geometric curve, making visualization a tool for doing and understanding algebra. Geometric shapes can be described by equations, making algebraic manipulation into a tool for geometric understanding, modeling and proof.

## Content Outline

### Congruence

### Similarity, Right Triangles, and Trigonometry

### Circles

### Expressing Geometric Properties with Equations

### Trigonometry of General Triangles

### Geometric Measurement and Dimension

### Modeling with Geometry

## Congruence

G-CO

1. Understand that two geometric figures are congruent if there is a sequence of rigid motions (rotations, reflections, translations) that carries one onto the other. This is the principle of superposition.
2. Understand that criteria for triangle congruence are ways to specify enough measures in a triangle to ensure that all triangles drawn with those measures are congruent.
3. Understand that criteria for triangle congruence (ASA, SAS, and SSS) can be established using rigid motions.
4. Understand that geometric diagrams can be used to test conjectures and identify logical errors in fallacious proofs.
5. Know and use (in reasoning and problem solving) definitions of angles, polygons, parallel, and perpendicular lines, rigid motions, parallelograms and rectangles.
6. Prove theorems about lines and angles. *Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; two lines parallel to a third are parallel to each other; points on a perpendicular bisector of a segment are exactly those equidistant from the segment's endpoints.*
7. Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180°, base angles of isosceles triangles are congruent, the triangle inequality, the longest side of a triangle faces the angle with the greatest measure and vice-versa, the exterior-angle inequality, and the segment joining midpoints of two sides of a triangle parallel to the third side and half the length.*
8. Use and prove properties of and relationships among special quadrilaterals: parallelogram, rectangle, rhombus, square, trapezoid and kite.
9. Characterize parallelograms in terms of equality of opposite sides, in terms of equality of opposite angles, and in terms of bisection of diagonals; characterize rectangles as parallelograms with equal diagonals.
10. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc). *Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.*
11. Construct an equilateral triangle, a square and a regular hexagon inscribed in a circle.
12. Use two-dimensional representations to transform figures and to predict the effect of translations, rotations, and reflections.
13. Use two-dimensional representations to transform figures and to predict the effect of dilations.

## Similarity, Right Triangles, and Trigonometry

G-SRT

1. Understand that dilating a line produces a line parallel to the original. (In particular, lines passing through the center of the dilation remain unchanged.)
2. Understand that the dilation of a given segment is parallel to the given segment and longer or shorter in the ratio given by the scale factor. A dilation leaves a segment unchanged if and only if the scale factor is 1.
3. Understand that the assumed properties of dilations can be used to establish the AA, SAS, and SSS criteria for similarity of triangles.
4. Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of sine, cosine, and tangent.
5. Understand that a line parallel to one side of a triangle divides the other two proportionally, and conversely.
6. Use triangle similarity criteria to solve problems and to prove relationships in geometric figures. *Include a proof of the Pythagorean theorem using triangle similarity.*
7. Use and explain the relationship between the sine and cosine of complementary angles.
8. Use sine, cosine, tangent, and the Pythagorean Theorem to solve right triangles[2] in applied problems.
9. STEM Give an informal explanation using successive approximation that a dilation of scale factor $r$ changes the length of a curve by a factor of $r$ and the area of a region by a factor of $r^2$.

## Circles

G-C

1. Understand that dilations can be used to show that all circles are similar.
2. Understand that there is a unique circle through three non-collinear points, and four circles tangent to three non-concurrent lines.

[2] A right triangle has five parameters, its three lengths and two acute angles. Given a length and any other parameter, "solving a right triangle" means finding the remaining three parameters.

3. Identify and define radius, diameter, chord, tangent, secant, and circumference.
4. Identify and describe relationships among angles, radii, and chords. *Include the relationship between central, inscribed and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.*
5. Determine the arc lengths and the areas of sectors of circles, using proportions.
6. STEM Construct a tangent line from a point outside a given circle to the circle.
7. STEM Prove and use theorems about circles, and use these theorems to solve problems involving:
    a. Symmetries of a circle
    b. Similarity of a circle to any other
    c. Tangent line, perpendicularity to a radius
    d. Inscribed angles in a circle, relationship to central angles, and equality of inscribed angles
    e. Properties of chords, tangents, and secants as an application of triangle similarity.

## Expressing Geometric Properties with Equations G-GPE

1. Understand that two lines with well-defined slopes are perpendicular if and only if the product of their slopes is equal to –1.
2. Understand that the equation of a circle can be found using its definition and the Pythagorean Theorem.
3. Understand that transforming the graph of an equation by reflecting in the axes, translating parallel to the axes, or applying a dilation in one of the coordinate directions corresponds to substitutions in the equation.
4. STEM Understand that an ellipse is the set of all points whose distances from two fixed points (the foci) are a constant sum. The graph of $x^2/a^2 + y^2/b^2 = 1$ is an ellipse with foci on one of the axes.
5. STEM Understand that a parabola is the set of points equidistant from a fixed point (the focus) and a fixed line (the directrix). The graph of any quadratic function is a parabola, and all parabolas are similar.
6. STEM Understand that the formula $A = \pi ab$ for the area of an ellipse can be derived from the formula for the area of a circle.★
7. Use the slope criteria for parallel and perpendicular lines to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).
8. Find the point on the segment between two given points that divides the segment in a given ratio.
9. Use coordinates to compute perimeters of polygons and areas for triangles and rectangles, e.g. using the distance formula.★
10. Decide whether a point with given coordinates lies on a circle defined by a given equation.
11. Use coordinates to prove simple geometric theorems algebraically. For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point $(1, \sqrt{3})$ lies on the circle centered at the origin and containing the point (0, 2).
12. Complete the square to find the center and radius of a circle given by an equation.
13. STEM Find an equation for an ellipse given in the coordinate plane with major and minor axes parallel to the coordinate axes.
14. STEM Calculate areas of ellipses to solve problems.★

## Trigonometry of General Triangles G-TGT

1. STEM Understand that the formula $A = ½\, ab \sin(C)$ for the area of a triangle can be derived by drawing an auxiliary line from a vertex perpendicular to the opposite side. Applying this formula in three different ways leads to the Law of Sines.
2. STEM Understand that the Law of Cosines generalizes the Pythagorean Theorem.
3. STEM Understand that the sine, cosine and tangent of the sum or difference of two angles can be expressed in terms of sine, cosine, and tangent of the angles themselves using the addition formulas.
4. STEM Understand that the Laws of Sines and Cosines embody the triangle congruence criteria, in that three pieces of information are usually sufficient to completely solve a triangle. Furthermore, these laws yield two possible solutions in the ambiguous case, illustrating that "Side-Side-Angle" is not a congruence criterion.
5. STEM Explain proofs of the Law of Sines and the Law of Cosines.

★ Standard with close connection to modeling.

6. STEM Use the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

## Geometric Measurement and Dimension — G-GMD

1. Understand that the area of a decomposed figure is the sum of the areas of its components and is independent of the choice of dissection.
2. STEM Understand that lengths of curves and areas of curved regions can be defined using the informal notion of limit.
3. STEM Understand that Cavalieri's principle allows one to understand volume formulas informally by visualizing volumes as stacks of thin slices.
4. Find areas of polygons by dissecting them into triangles.
5. Explain why the volume of a cylinder is the area of the base times the height, using informal arguments.
6. For a pyramid or a cone, give a heuristic argument to show why its volume is one-third of its height times the area of its base.
7. Apply formulas and solve problems involving volume and surface area of right prisms, right circular cylinders, right pyramids, cones, spheres and composite figures.
8. STEM Identify cross-sectional shapes of slices of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.
9. STEM Use the behavior of length and area under dilations to show that the circumference of a circle is proportional to the radius and the area of a circle is proportional to the square of the radius. Identify the relation between the constants of proportionality with an informal argument involving dissection and recomposition of a circle into an approximate rectangle.

## Modeling with Geometry — G-MG

1. Understand that models of objects and structures can be built from a library of standard shapes; a single kind of shape can model seemingly different objects.★
2. Use geometric shapes, their measures and their properties to describe objects (e.g., modeling a tree trunk or a human torso or as a cylinder).★
3. Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).★
4. Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy constraints or minimize cost; working with typographic grid systems based on ratios).★

★ Standard with close connection to modeling.

# Glossary

**Addition and subtraction within 10, 20, or 100.** Addition or subtraction of whole numbers with whole number answers, and with sum or minuend at most 10, 20, or 100. Example: $8 + 2 = 10$ is an addition within 10, $14 - 5 = 9$ is a subtraction within 20, and $55 - 18 = 37$ is a subtraction within 100.

**Additive inverses.** Two numbers whose sum is 0 are additive inverses of one another. Example: $3/4$ and $-3/4$ are additive inverses of one another because $3/4 + (-3/4) = (-3/4) + 3/4 = 0$.

**Box plot.** A method of visually displaying a distribution of data values by using the median, quartiles, and extremes of the data set. A box shows the middle 50% of the data.[3]

**Complex fraction.** A fraction $A/B$ where $A$ and/or $B$ are fractions.

**Congruent.** Two plane or solid figures are congruent if one can be obtained from the other by a sequence of rigid motions (rotations, reflections, and translations).

**Counting on.** A strategy for finding the number of objects in a group without having to count every member of the group. For example, if a stack of books is known to have 8 books and 3 more books are added to the top, it is not necessary to count the stack all over again; one can find the total by *counting on*—pointing to the top book and saying "eight," following this with "nine, ten, eleven. There are eleven books now."

**Decade word.** A word referring to a single-digit multiple of ten, as in *twenty*, *thirty*, *forty*, etc.

**Dot plot.** A method of visually displaying a distribution of data values where each data value is shown as a dot or mark above a number line. Also known as a line plot.[4]

**Dilation.** A transformation that moves each point along the ray through the point emanating from a fixed center, and multiplies distances from the center by a common scale factor.

**Empirical probability model.** A probablity model based on a data set for a random process in which the probability of a particular type or category of outcome equals the percentage of data points included in the category. Example: If a coin is tossed 10 times and 4 of the tosses are Heads, then the empirical probability of Heads in the empirical probability model is $4/10$ (equivalently 0.4 or 40%).

**Equivalent fractions.** Two fractions $a/b$ and $c/d$ that represent the same number.

**Expanded form.** A multidigit number is expressed in expanded form when it is written as a sum of single-digit multiples of powers of ten. For example, $643 = 600 + 40 + 3$.

**First quartile.** For a data set with median $M$, the first quartile is the median of the data values less than $M$. Example: For the data set {1, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the first quartile is 6.[5] *See also* median, third quartile, interquartile range.

**Fraction.** A number expressible in the form $a/b$ where $a$ is a whole number and $b$ is a positive whole number. (The word *fraction* in these standards always refers to a nonnegative number.) *See also* rational number.

**Independently combined probability models.** Two probability models are said to be combined independently if the probability of each ordered pair in the combined model equals the product of the original probabilities of the two individual outcomes in the ordered pair.

**Integer.** A number expressible in the form $a$ or $-a$ for some whole number $a$.

**Interquartile Range.** A measure of variation in a set of numerical data, the interquartile range is the distance between the first and third quartiles of the data set. Example: For the data set {1, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the interquartile range is $15 - 6 = 9$. *See also* first quartile, third quartile.

**Laws of arithmetic.** See Table 3 in this Glossary.

**Line plot.** See dot plot.

**Mean.** A measure of center in a set of numerical data, computed by adding the values in a list and then dividing by the number of values in the list.[6] Example: For the data set {1, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the mean is 21.

**Mean absolute deviation.** A measure of variation in a set of numerical data, computed by adding the distances between each data value and the mean, then dividing by the number of data values. Example: For the data set {2, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the mean absolute deviation is 20.

**Median.** A measure of center in a set of numerical data. The median of a list of values is the value appearing at the center of a sorted version of the list—or the mean of the two central values, if the list contains an even number of values. Example: For the data set {2, 3, 6, 7, 10, 12, 14, 15, 22, 90}, the median is 11.

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3 Adapted from Wisconsin Department of Public Instruction, http://dpi.wi.gov/standards/mathglos.html, accessed March 2, 2010.

4 Adapted from Wisconsin Department of Public Instruction, *op. cit.*.

5 Many different methods for computing quartiles are in use. The method defined here is sometimes called the Moore and McCabe method. See Langford, E., "Quartiles in Elementary Statistics," *Journal of Statistics Education* Volume 14, Number 3 (2006),

6 To be more precise, this defines the *arithmetic mean*.

**Multiplication and division within 100.** Multiplication or division of whole numbers with whole number answers, and with product or dividend at most 100. Example: $72 \div 8 = 9$.

**Multiplicative inverses.** Two numbers whose product is 1 are multiplicative inverses of one another. Example: $3/4$ and $4/3$ are multiplicative inverses of one another because $3/4 \times 4/3 = 4/3 \times 3/4 = 1$.

**Properties of equality.** See Table 4 in this Glossary.

**Properties of inequality.** See Table 5 in this Glossary.

**Properties of operations.** Associativity and commutativity of addition and multiplication, distributivity of multiplication over addition, the additive identity property of 0, and the multiplicative identity property of 1.See Table 3 in this Glossary.

**Probability.** A number between 0 and 1 used to quantify likelihood for processes that have uncertain outcomes (such as tossing a coin, selecting a person at random from a group of people, tossing a ball at a target, testing for a medical condition).

**Rational number.** A number expressible in the form $a/b$ or $-a/b$ for some fraction $a/b$. The rational numbers include the integers.

**Related fractions.** Two fractions are said to be related if one denominator is a factor of the other.[7]

**Rigid motion.** A transformation of points in space consisting of one or more translations, reflections, and/or rotations. Rigid motions are here assumed to preserve distances and angle measures.

**Sample space.** In a probability model for a random process, a list of the individual outcomes that are to be considered.

**Scatter plot.** A graph in the coordinate plane representing a set of bivariate data. For example, the heights and weights of a group of people could be displayed on a scatter plot.[8]

**Similarity transformation.** A rigid motion followed by a dilation.

**Tape diagrams.** Drawings that look like a segment of tape, used to illustrate number relationships. Also known as strip diagrams, bar models or graphs, fraction strips, or length models.

**Teen number.** A whole number that is greater than or equal to 11 and less than or equal to 19.

**Third quartile.** For a data set with median $M$, the third quartile is the median of the data values greater than $M$. Example: For the data set $\{2, 3, 6, 7, 10, 12, 14, 15, 22, 120\}$, the third quartile is 15. *See also* median, first quartile, interquartile range.

**Uniform probability model.** A probability model in which the individual outcomes all have the same probability ($1/N$ if there are $N$ individual outcomes in the sample space). If a given type of outcome consists of $M$ individual outcomes, then the probability of that type of outcome is $M/N$. Example: if a uniform probability model is used to model the process of randomly selecting a person from a class of 32 students, and if 8 of the students are left-handed, then the probability of randomly selecting a left-handed student is $8/32$ (equivalently $1/4$, 0.25 or 25%).

**Whole numbers.** The numbers 0, 1, 2, 3, ....

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[7] See Ginsburg, Leinwand and Decker (2009), *Informing Grades 1-6 Mathematics Standards Development: What Can Be Learned from High-Performing Hong Kong, Korea, and Singapore?*, Table A1, p. A-5, grades 3 and 4.

[8] Adapted from Wisconsin Department of Public Instruction, *op. cit.*.

TABLE 1. Common addition and subtraction situations.[9]

| | Result Unknown | Change Unknown | Start Unknown |
|---|---|---|---|
| Add to | Two bunnies sat on the grass. Three more bunnies hopped there. How many bunnies are on the grass now? $2 + 3 = ?$ | Two bunnies were sitting on the grass. Some more bunnies hopped there. Then there were five bunnies. How many bunnies hopped over to the first two? $2 + ? = 5$ | Some bunnies were sitting on the grass. Three more bunnies hopped there. Then there were five bunnies. How many bunnies were on the grass before? $? + 3 = 5$ |
| Take from | Five apples were on the table. I ate two apples. How many apples are on the table now? $5 - 2 = ?$ | Five apples were on the table. I ate some apples. Then there were three apples. How many apples did I eat? $5 - ? = 3$ | Some apples were on the table. I ate two apples. Then there were three apples. How many apples were on the table before? $? - 2 = 3$ |

| | Total Unknown | Addend Unknown | Both Addends Unknown[10] |
|---|---|---|---|
| Put Together/ Take Apart[11] | Three red apples and two green apples are on the table. How many apples are on the table? $3 + 2 = ?$ | Five apples are on the table. Three are red and the rest are green. How many apples are green? $3 + ? = 5,\ 5 - 3 = ?$ | Grandma has five flowers. How many can she put in her red vase and how many in her blue vase? $5 = 0 + 5, 5 = 5 + 0$ $5 = 1 + 4, 5 = 4 + 1$ $5 = 2 + 3, 5 = 3 + 2$ |

| | Difference Unknown | Bigger Unknown | Smaller Unknown |
|---|---|---|---|
| Compare[12] | ("How many more?" version): Lucy has two apples. Julie has five apples. How many more apples does Julie have than Lucy? ("How many fewer?" version): Lucy has two apples. Julie has five apples. How many fewer apples does Lucy have than Julie? $2 + ? = 5,\ 5 - 2 = ?$ | (Version with "more"): Julie has three more apples than Lucy. Lucy has two apples. How many apples does Julie have? (Version with "fewer"): Lucy has 3 fewer apples than Julie. Lucy has two apples. How many apples does Julie have? $2 + 3 = ?,\ 3 + 2 = ?$ | (Version with "more"): Julie has three more apples than Lucy. Julie has five apples. How many apples does Lucy have? (Version with "fewer"): Lucy has 3 fewer apples than Julie. Julie has five apples. How many apples does Lucy have? $5 - 3 = ?,\ ? + 3 = 5$ |

[9] Adapted from Box 2-4 of National Research Council (2009, op. cit., pp. 32, 33).

[10] These *take apart* situations can be used to show all the decompositions of a given number. The associated equations, which have the total on the left of the equal sign, help children understand that the = sign does not always mean *makes* or *results in* but always does mean *is the same number as*.

[11] Either addend can be unknown, so there are three variations of these problem situations. Both Addends Unknown is a productive extension of this basic situation especially for small numbers less than or equal to 10.

[12] For the Bigger Unknown or Smaller Unknown situations, one version directs the correct operation (the version using *more* for the bigger unknown and using *less* for the smaller unknown). The other versions are more difficult.

TABLE 2. Common multiplication and division situations.[13]

| | Unknown Product | Group Size Unknown ("How many in each group?" Division) | Number of Groups Unknown ("How many groups?" Division) |
|---|---|---|---|
| | $3 \times 6 = ?$ | $3 \times ? = 18$ and $18 \div 3 = ?$ | $? \times 6 = 18$ and $18 \div 6 = ?$ |
| Equal Groups | There are 3 bags with 6 plums in each bag. How many plums are there in all?<br>*Measurement example.* You need 3 lengths of string, each 6 inches long. How much string will you need altogether? | If 18 plums are shared equally into 3 bags, then how many plums will be in each bag?<br>*Measurement example.* You have 18 inches of string, which you will cut into 3 equal pieces. How long will each piece of string be? | If 18 plums are to be packed 6 to a bag, then how many bags are needed?<br>*Measurement example.* You have 18 inches of string, which you will cut into pieces that are 6 inches long. How many pieces of string will you have? |
| Arrays,[14] Area[15] | There are 3 rows of apples with 6 apples in each row. How many apples are there?<br>*Area example.* What is the area of a 3 cm by 6 cm rectangle? | If 18 apples are arranged into 3 equal rows, how many apples will be in each row?<br>*Area example.* A rectangle has area 18 square centimeters. If one side is 3 cm long, how long is a side next to it? | If 18 apples are arranged into equal rows of 6 apples, how many rows will there be?<br>*Area example.* A rectangle has area 18 square centimeters. If one side is 6 cm long, how long is a side next to it? |
| Compare | A blue hat costs $6. A red hat costs 3 times as much as the blue hat. How much does the red hat cost?<br>*Measurement example.* A rubber band is 6 cm long. How long will the rubber band be when it is stretched to be 3 times as long? | A red hat costs $18 and that is 3 times as much as a blue hat costs. How much does a blue hat cost?<br>*Measurement example.* A rubber band is stretched to be 18 cm long and that is 3 times as long as it was at first. How long was the rubber band at first? | A red hat costs $18 and a blue hat costs $6. How many times as much does the red hat cost as the blue hat?<br>*Measurement example.* A rubber band was 6 cm long at first. Now it is stretched to be 18 cm long. How many times as long is the rubber band now as it was at first? |
| General | $a \times b = ?$ | $a \times ? = p$ and $p \div a = ?$ | $? \times b = p$ and $p \div b = ?$ |

[13] The first examples in each cell are examples of discrete things. These are easier for students and should be given before the measurement examples.

[14] The language in the array examples shows the easiest form of array problems. A harder form is to use the terms rows and columns: The apples in the grocery window are in 3 rows and 6 columns. How many apples are in there? Both forms are valuable.

[15] Area involves arrays of squares that have been pushed together so that there are no gaps or overlaps, so array problems include these especially important measurement situations.

TABLE 3. The laws of arithmetic, including the properties of operations (identified with °). Here $a$, $b$ and $c$ stand for arbitrary numbers in a given number system. The laws of arithmetic apply to the rational number system, the real number system, and the complex number system.

| | |
|---|---|
| °*Associative law of addition* | $(a + b) + c = a + (b + c)$ |
| °*Commutative law of addition* | $a + b = b + a$ |
| °*Additive identity property of 0* | $a + 0 = 0 + a = a$ |
| *Existence of additive inverses* | For every $a$ there exists $-a$ so that $a + (-a) = (-a) + a = 0$. |
| °*Associative law of multiplication* | $(a \times b) \times c = a \times (b \times c)$ |
| °*Commutative law of multiplication* | $a \times b = b \times a$ |
| °*Multiplicative identity property of 1* | $a \times 1 = 1 \times a = a$ |
| *Existence of multiplicative inverses* | For every $a \neq 0$ there exists $1/a$ so that $a \times 1/a = 1/a \times a = 1$. |
| °*Distributive law of multiplication over addition* | $a \times (b + c) = a \times b + a \times c$ |

TABLE 4. The properties of equality. Here $a$, $b$ and $c$ stand for arbitrary numbers in the rational, real, or complex number systems.

| | |
|---|---|
| *Reflexive property of equality* | $a = a$ |
| *Symmetric property of equality* | If $a = b$, then $b = a$. |
| *Transitive property of equality* | If $a = b$ and $b = c$, then $a = c$. |
| *Addition property of equality* | If $a = b$, then $a + c = b + c$. |
| *Subtraction property of equality* | If $a = b$, then $a - c = b - c$. |
| *Multiplication property of equality* | If $a = b$, then $a \times c = b \times c$. |
| *Division property of equality* | If $a = b$ and $c \neq 0$, then $a \div c = b \div c$. |
| *Substitution property of equality* | If $a = b$, then $b$ may be substituted for $a$ in any expression containing $a$. |

TABLE 5. The properties of inequality. Here $a$, $b$ and $c$ stand for arbitrary numbers in the rational or real number systems.

Exactly one of the following is true: $a < b$, $a = b$, $a > b$.

If $a > b$ and $b > c$ then $a > c$.

If $a > b$, then $b < a$.

If $a > b$, then $-a < -b$.

If $a > b$, then $a \pm c > b \pm c$.

If $a > b$ and $c > 0$, then $a \times c > b \times c$.

If $a > b$ and $c < 0$, then $a \times c < b \times c$.

If $a > b$ and $c > 0$, then $a \div c > b \div c$.

If $a > b$ and $c < 0$, then $a \div c < b \div c$.

# Sample of Works Consulted


Existing state standards documents.
Research summaries and briefs provided to the Working Group by researchers.
Mathematics documents from: Alberta, Canada; Belgium; China; Chinese Taipei; Denmark; England; Finland; Hong Kong; India; Ireland; Japan; Korea, New Zealand, Singapore; Victoria (British Columbia).
Adding it Up: Helping Children Learn Mathematics. National Research Council, Mathematics Learning Study Committee, 2001.
Benchmarking for Success: Ensuring U.S. Students Receive a World-Class Education. National Governors Association, Council of Chief State School Officers, and Achieve, Inc., 2008.
*Crossroads in Mathematics* (1995) and *Beyond Crossroads* (2006). American Mathematical Association of Two-Year Colleges (AMATYC).
*Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence.* National Council of Teachers of Mathematics, 2006.
*Focus in High School Mathematics: Reasoning and Sense Making.* National Council of Teachers of Mathematics. Reston, VA: NCTM.
*Foundations for Success: The Final Report of the National Mathematics Advisory Panel.* U.S. Department of Education: Washington, DC, 2008.
*Guidelines for Assessment and Instruction in Statistics Education (GAISE) Report: A PreK-12 Curriculum Framework.*
*How People Learn: Brain, Mind, Experience, and School.* Bransford, J.D., Brown, A.L., and Cocking, R.R., eds. Committee on Developments in the Science of Learning, Commission on Behavioral and Social Sciences and Education, National Research Council, 1999.
*Mathematics and Democracy, The Case for Quantitative Literacy*, Steen, L.A. (ed.). National Council on Education and the Disciplines, 2001.
*Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity.* Cross, C.T., Woods, T.A., and Schweingruber, S., eds. Committee on Early Childhood Mathematics, National Research Council, 2009.
*The Opportunity Equation: Transforming Mathematics and Science Education for Citizenship and the Global Economy.* The Carnegie Corporation of New York and the Institute for Advanced Study, 2009. Online: http://www.opportunityequation.org/
*Principles and Standards for School Mathematics.* National Council of Teachers of Mathematics, 2000.
*The Proficiency Illusion.* Cronin, J., Dahlin, M., Adkins, D., and Kingsbury, G.G.; foreword by C.E. Finn, Jr., and M. J. Petrilli. Thomas B. Fordham Institute, 2007.
*Ready or Not: Creating a High School Diploma That Counts.* American Diploma Project, 2004.
*A Research Companion to Principles and Standards for School Mathematics.* National Council of Teachers of Mathematics, 2003.
*Sizing Up State Standards 2008.* American Federation of Teachers, 2008.
*A Splintered Vision: An Investigation of U.S. Science and Mathematics Education.* Schmidt, W.H., McKnight, C.C., Raizen, S.A., et al. U.S. National Research Center for the Third International Mathematics and Science Study, Michigan State University, 1997.
*Stars By Which to Navigate? Scanning National and International Education Standards in 2009.* Carmichael, S.B., W.S. Wilson, Finn, Jr., C.E., Winkler, A.M., and Palmieri, S. Thomas B. Fordham Institute, 2009.
Blum, W., Galbraith, P. L., Henn, H-W. and Niss, M. (Eds) *Applications and Modeling in Mathematics Education*, ICMI Study 14. Amsterdam: Springer.
Cobb and Moore, "Mathematics, Statistics, and Teaching," *Amer. Math. Monthly* 104(9), pp. 801-823, 1997.
Conley, D.T. *Knowledge and Skills for University Success*, 2008.
Conley, D.T. *Toward a More Comprehensive Conception of College Readiness*, 2007.
Cuoco, A., Goldenberg, E. P., and Mark, J., "Habits of Mind: An Organizing Principle for a Mathematics Curriculum," *Journal of Mathematical Behavior*, 15(4), 375-402, 1996.
Ginsburg, A., Leinwand, S., and Decker, K., "Informing Grades 1-6 Standards Development: What Can Be Learned from High-Performing Hong Kong, Korea, and Singapore?" American Institutes for Research, 2009.
Ginsburg et al., "What the United States Can Learn From Singapore's World-Class Mathematics System (and what Singapore can learn from the United States)," American Institutes for Research, 2005.
Ginsburg et al., "Reassessing U.S. International Mathematics Performance: New Findings from the 2003 TIMMS and PISA," American Institutes for Research, 2005.
Harel, G., "What is Mathematics? A Pedagogical Answer to a Philosophical Question," in R. B. Gold and R. Simons (Eds.), *Current Issues in the Philosophy of Mathematics from the Perspective of Mathematicians.* Mathematical Association of America, 2008.
Howe, R., "From Arithmetic to Algebra."
Howe, R., "Starting Off Right in Arithmetic," http://math.arizona.edu/~ime/2008-09/MIME/BegArith.pdf.
Jordan, N. C., Kaplan, D., Ramineni, C., and Locuniak, M. N., "Early math matters: kindergarten number competence and later mathematics outcomes," *Dev. Psychol.* 45, 850–867, 2009.
Kilpatrick, J., Mesa, V., and Sloane, F., "U.S. Algebra Performance in an International Context," in Loveless (ed.), *Lessons Learned: What International Assessments Tell Us About Math Achievement.* Washington, D.C.: Brookings Institution Press, 2007.
Leinwand, S., and Ginsburg, A., "Measuring Up: How the Highest Performing state (Massachusetts) Compares to the Highest Performing Country (Hong Kong) in Grade 3 Mathematics," American Institutes for Research, 2009.
Niss, M., "Quantitative Literacy and Mathematical Competencies," in *Quantitative Literacy: Why Numeracy Matters for Schools and Colleges*, Madison, B. L., and Steen, L.A. (eds.), National Council on Education and the Disciplines. Proceedings of the National Forum on Quantitative Literacy held at the National Academy of Sciences in Washington, D.C., December 1-2, 2001.
Reys, B. (ed.), *The Intended Mathematics Curriculum as Represented in State-Level Curriculum Standards: Consensus or Confusion?* IAP-Information Age Publishing, 2006.
Schmidt, W., Houang, R., and Cogan, L., "A Coherent Curriculum: The Case of Mathematics," *American Educator*, Summer 2002, p. 4.
Schmidt, W.H. and Houang, R.T., "Lack of Focus in the Intended Mathematics Curriculum: Symptom or Cause?" in Loveless (ed.), *Lessons Learned: What International Assessments Tell Us About Math Achievement.* Washington, D.C.: Brookings Institution Press, 2007.
Wu, H., "Fractions, decimals, and rational numbers," 2007, http://math.berkeley.edu/~wu/ (March 19, 2008).
Wu, H., "Lecture Notes for the 2009 Pre-Algebra Institute," September 15, 2009.
Wu, H., "Preservice professional development of mathematics Teachers," http://math.berkeley.edu/~wu/pspd2.pdf.
Massachusetts Department of Education. Progress Report of the Mathematics Curriculum Framework Revision Panel, Massachusetts Department of Elementary and Secondary Education, 2009. www.doe.mass.edu/boe/docs/0509/item5_report.pdf.
ACT College Readiness Benchmarks™
ACT College Readiness Standards™
ACT National Curriculum Survey™
Adelman, C. *The Toolbox Revisited: Paths to Degree Completion From High School Through College*, 2006.
*Advanced Placement Calculus, Statistics and Computer Science Course Descriptions. May 2009, May 2010.* College Board, 2008.
*Aligning Postsecondary Expectations and High School Practice: The Gap Defined* (ACT: Policy Implications of the ACT National Curriculum Survey Results 2005-2006).
*Condition of Education, 2004: Indicator 30, Top 30 Postsecondary Courses*, U.S. Department of Education, 2004.
*Condition of Education, 2007: High School Course-Taking.* U.S. Department of Education, 2007.
*Crisis at the Core: Preparing All Students for College and Work*, ACT.
Achieve, Inc., Florida Postsecondary Survey, 2008.
Golfin, Peggy, et. al. CNA Corporation. *Strengthening Mathematics at the Postsecondary Level: Literature Review and Analysis*, 2005.
Camara, W.J., Shaw, E., and Patterson, B. (June 13, 2009). First Year English and Math College Coursework. College Board: New York, NY (Available from authors).
CLEP Precalculus Curriculum Survey: Summary of Results. The College Board, 2005.
College Board Standards for College Success: Mathematics and Statistics. College Board, 2006.
Miller, G.E., Twing, J., and Meyers, J. "Higher Education Readiness Component (HERC) Correlation Study." Austin, TX: Pearson.
*On Course for Success: A Close Look at Selected High School Courses That Prepare All Students for College and Work*, ACT.

*Out of Many, One: Towards Rigorous Common Core Standards from the Ground Up*. Achieve, 2008.
*Ready for College and Ready for Work: Same or Different?* ACT.
Rigor at Risk: Reaffirming Quality in the High School Core Curriculum, ACT.
*The Forgotten Middle: Ensuring that All Students Are on Target for College and Career Readiness before High School*, ACT.
Achieve, Inc., Virginia Postsecondary Survey, 2004.
ACT Job Skill Comparison Charts
Achieve, Mathematics at Work, 2008.
*The American Diploma Project Workplace Study*. National Alliance of Business Study, 2002.
Carnevale, Anthony and Desrochers, Donna. *Connecting Education Standards and Employment: Course-taking Patterns of Young Workers*, 2002.
Colorado Business Leaders Top Skills, 2006.
*Hawai'i Career Ready Study: access to living wage careers from high school*, 2007.
States' Career Cluster Initiative. *Essential Knowledge and Skill Statements*, 2008.
ACT WorkKeys Occupational Profiles™
Program for International Student Assessment (PISA), 2006.
Trends in International Mathematics and Science Study (TIMSS), 2007.
International Baccalaureate, Mathematics Standard Level, 2006.
University of Cambridge International Examinations: General Certificate of Secondary Education in Mathematics, 2009.
EdExcel, General Certificate of Secondary Education, Mathematics, 2009.
Blachowicz, Camille, and Peter Fisher. "Vocabulary Instruction." In *Handbook of Reading Research*, Volume III, edited by Michael Kamil, Peter Mosenthal, P. David Pearson, and Rebecca Barr, pp. 503-523. Mahwah, NJ: Lawrence Erlbaum Associates, 2000.
Gándara, Patricia, and Frances Contreras. *The Latino Education Crisis: The Consequences of Failed Social Policies*. Cambridge, Ma: Harvard University Press, 2009.
Moschkovich, Judit N. "Supporting the Participation of English Language Learners in Mathematical Discussions." *For the Learning of Mathematics* 19 (March 1999): 11-19.
Moschkovich, J. N. (in press). Language, culture, and equity in secondary mathematics classrooms. To appear in F. Lester & J. Lobato (Ed.), Teaching and Learning Mathematics: Translating Research to the Secondary Classroom, Reston, VA: NCTM.
Moschkovich, Judit N. "Examining Mathematical Discourse Practices," *For the Learning of Mathematics* 27 (March 2007): 24-30.
Moschkovich, Judit N. "Using Two Languages when Learning Mathematics: How Can Research Help Us Understand Mathematics Learners Who Use Two Languages?" *Research Brief and Clip*, National Council of Teachers of Mathematics, 2009 http://www.nctm.org/uploadedFiles/Research_News_and_Advocacy/Research/Clips_and_Briefs/Research_brief_12_Using_2.pdf. (accessed November 25, 2009).
Moschkovich, J.N. (2007) Bilingual Mathematics Learners: How views of language, bilingual learners, and mathematical communication impact instruction. In N. Nasir and P. Cobb (Eds.), *Diversity, Equity, and Access to Mathematical Ideas*. New York: Teachers College Press, 89-104.
Schleppegrell, M.J. (2007). The linguistic challenges of mathematics teaching and learning: A research review. *Reading & Writing Quarterly*, 23:139-159.
Individuals with Disabilities Education Act (IDEA), 34 CFR §300.34 (a). (2004).
Individuals with Disabilities Education Act (IDEA), 34 CFR §300.39 (b)(3). (2004).
Office of Special Education Programs, U.S. Department of Education. "IDEA Regulations: Identification of Students with Specific Learning Disabilities," 2006.
Thompson, S. J., Morse, A.B., Sharpe, M., and Hall, S., "Accommodations Manual: How to Select, Administer and Evaluate Use of Accommodations and Assessment for Students with Disabilities," 2nd Edition. Council of Chief State School Officers, 2005.

# *Resource and Supporting Materials*

# *Grade 4 Unit 6 Sample*

As requested, attached please find resources with supporting materials.

You will find the following:

- Unit content standards and mathematical practice standards related to this unit
- A one week overview in the unit mapping the progression of lessons
- Sample of lessons L11, L12, and L15 with supporting materials
  Samples include:
    - L11 addressing the content standard that angles are additive with support materials
    - L12 is additional practice with measuring angles in a context and reinforces angle measure as additive
    - L15 cites resources used in the lesson development. (TSCM) *Teaching Student Centered Mathematics,* (BLM) Black line master, (SF) *Scott Foresman* textbook (All are teacher resources currently in the district.)
- Differentiation suggestions are included within the lesson structure.

# Fairfield Public Schools

# Balanced Mathematics Instructional Model

## Grade 4

## Unit Six

### Geometry and Measurement

Common Core State Standards
Grade 4
Unit 6: Geometry and Measurement

**Measurement and Data**

**Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit.**

4.MD.3 Apply the area and perimeter formulas for rectangles in real world and mathematical problems. *For example, find the width of a rectangular room given the area of the flooring and the length, by viewing the area formula as a multiplication equation with an unknown factor.*

**Geometric measurement: understand concepts of angle and measure angles.**

4.MD.5 Recognize angles as geometric shapes that are formed wherever two rays share a common endpoint, and understand concepts of angle measurement:

a. An angle is measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the two rays intersect the circle. An angle that turns through 1/360 of a circle is called a "one-degree angle," and can be used to measure angles.

b. An angle that turns through $n$ one-degree angles is said to have an angle measure of $n$ degrees.

4.MD.6 Measure angles in whole-number degrees using a protractor. Sketch angles of specified measure.

4.MD.7 Recognize angle measure as additive. When an angle is decomposed into non-overlapping parts, the angle measure of the whole is the sum of the angle measures of the parts. Solve addition and subtraction problems to find unknown angles on a diagram in real world and mathematical problems, e.g., by using an equation with a symbol for the unknown angle measure.

**Geometry**

**Draw and identify lines and angles, and classify shapes by properties of their lines and angles.**

4.G.1 Draw points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Identify these in two-dimensional figures.

4.G.2 Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles.

4.G.3 Recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts. Identify line-symmetric figures and draw lines of symmetry.

## Grade Four Standards for Mathematical Practice

The K-12 Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. This page gives examples of what the practice standards look like at the specified grade level.

| *Standards* | *Explanations and Examples* |
|---|---|
| Students are expected to:<br>**1. Make sense of problems and persevere in solving them.** | In fourth grade, students know that doing mathematics involves solving problems and discussing how they solved them. Students explain to themselves the meaning of a problem and look for ways to solve it. Fourth graders may use concrete objects or pictures to help them conceptualize and solve problems. They may check their thinking by asking themselves, "Does this make sense?" They listen to the strategies of others and will try different approaches. They often will use another method to check their answers. |
| Students are expected to:.<br>**2. Reason abstractly and quantitatively.** | Fourth graders should recognize that a number represents a specific quantity. They connect the quantity to written symbols and create a logical representation of the problem at hand, considering both the appropriate units involved and the meaning of quantities. They extend this understanding from whole numbers to their work with fractions and decimals. Students write simple expressions that record calculations with numbers and represent or round numbers using place value concepts. |
| Students are expected to:<br>**3. Construct viable arguments and critique the reasoning of others.** | In fourth grade, students may construct arguments using concrete referents such as objects, pictures, and drawings. They explain their thinking and make connections between models and equations. They refine their mathematical communication skills as they participate in mathematical discussions involving questions like "How did you get that?" and "Why is that true?" They explain their thinking to others and respond to others' thinking. |
| Students are expected to:<br>**4. Model with mathematics.** | Students experiment with representing problem situations in multiple ways, including numbers, words (mathematical language), drawing pictures, using objects, making a chart, list, or graph, creating equations, etc. Students need opportunities to connect the different representations and explain the connections. They should be able to use all of these representations as needed. Fourth graders should evaluate their results in the context of the situation and reflect on whether the results make sense. |
| Students are expected to:<br>**5. Use appropriate tools strategically.** | Fourth graders consider the available tools (including estimation) when solving a mathematical problem and decide when certain tools might be helpful. For instance, they may use graph paper or a number line to represent and compare decimals and protractors to measure angles. They use other measurement tools to understand the relative size of units within a system and express measurements given in larger units in terms of smaller units. |
| Students are expected to:<br>**6. Attend to precision.** | As fourth graders develop their mathematical communication skills, they try to use clear and precise language in their discussions with others and in their own reasoning. They are careful about specifying units of measure and state the meaning of the symbols they choose. For instance, they use appropriate labels when creating a line plot. |
| Students are expected to:<br>**7. Look for and make use of structure.** | In fourth grade, students look closely to discover a pattern or structure. For instance, students use properties of operations to explain calculations (partial products model). They relate representations of counting problems such as tree diagrams and arrays to the multiplication principal of counting. They generate number or shape patterns that follow a given rule. |
| Students are expected to:<br>**8. Look for and express regularity in repeated reasoning.** | Students in fourth grade should notice repetitive actions in computation to make generalizations. Students use models to explain calculations and understand how algorithms work. They also use models to examine patterns and generate their own algorithms. For example, students use visual fraction models to write equivalent fractions. |

| Lesson #11 | Lesson #12 | Lesson #13 | Lesson #14 | Lesson #15 |
|---|---|---|---|---|
| **Teaching Point**<br>Angle measure is additive.<br><br>**Mini-Lesson:**<br>Model for students that the sum of two angles can add up to 90° and 180°.Tell the students they will be looking for missing angles based on the sum of the total angles<br><br>**APS:**<br>Students work in pairs to find the missing angles on Missing Angles Sheet I, II, & III. Missing angle sheet III is worthy of discussion but may be a stretch for some students. The purpose is to help students think about adding angles to find other angles rather than using a protractor to measure every angle.<br><br>**Assessment Point:**<br>Are students using the protractor to appropriately measure angles? Are students thinking about combined angle measures as additive?<br><br>**Focus question:**<br>What is the missing angle?<br>What strategy did you use to find it? | **Teaching Point**<br>Provide students with an opportunity to practice measuring angles in a context using a protractor.<br><br>**Mini-Lesson:**<br>Explain how to conduct an experiment where students bounce a ball of an object and measure both its approach angle and the reflected angle.<br><br>**APS:**<br>Students work in threes to bounce a ball off the wall and mark and measure the approaching angle and reflected angle. They then measure the angles and compare them and record their findings to share with the class.<br><br>**Assessment Point:**<br>Are students able to appropriately use and read the protractor?<br>Do student understand angle measure as a measure of rotation or turn?<br><br>**Focus question:**<br>If the ball hits the wall at an angle, at what angle does the ball bounce off the wall? | **Teaching Point**<br>Finding missing angles using 360°<br><br>**Mini-Lesson:**<br>Help students make the connection that the sum of all angles in a full rotation is360°. Explain the "pie chart problem" so that all students are clear about the task. It may be helpful to provide the class with a T-chart.<br><br>**APS:**<br>Students work with a partner to solve the Pie Chart Problem.<br><br>**Assessment Point:**<br>Are students thinking of 360° as additive?<br><br>**Focus question:**<br>1. What fraction of the total budget went to transportation?<br>2. What is the total amount of money she budgeted for the trip?<br>3. What is the measurement of the angle for the miscellaneous things?<br>4. How much money did she budget for miscellaneous things?<br>5. How much of the total budget did she spend on the cabin rental? | **Teaching Point**<br>Angles are additive<br><br>**Mini-Lesson:**<br>Model for students how angles of polygons are additive.<br><br>**APS:**<br>Students work with their partner to investigate whether the sum of interior angles is consistent for quadrilaterals and triangles.<br><br>**Assessment Point:**<br>Do students understand angels as additive?<br><br>**Focus question:**<br>What is the sum of interior angles of a quadrilateral?<br>What is the sum of the interior angles of a triangle?<br>Are the sums different?, the same?, is there a relationship? Why or why not?<br>Is the sum the same or different for all triangles, parallelograms, (other regular or irregular shapes)? | **Teaching Point**<br>Sort triangles and classify triangles:<br>Defining types of triangles: equilateral triangle, Isosceles triangle, Right triangle.<br><br>**Mini-Lesson:**<br>Define triangles and present students with a variety of triangles to sort and classify.<br><br>**APS:**<br>Students work collaboratively to sort and categorize triangles into three types.<br><br>**Assessment Point:**<br>Note whether students are grouping triangle based on appearance or base on attributes.<br><br>**Focus question:**<br>What are the different ways you can sort triangles into 3 different groups so that no triangle belongs to two groups? |

## Fairfield Public Schools Balanced Math Instructional Model

| | | |
|---|---|---|
| | **Materials:** "Missing Angles" SI, SII, SIII, protractors | |
| | **Fluency Work (1-5 min.)** | Make a T-chart. On one side label it "number of squares". On the other label it "number of sides." Begin to fill in the chart 1 / 4, 2 / 8, 3 / ?, 5 / ?, ? / 64, $n^{th}$ / ?. Ask students to express the rule in terms of both number of sides and the inverse, number of squares. Then use a different sided shape and repeat. |
| (5-10 min.) | **Teaching Point** | Angle measure is additive.<br>When an angle is decomposed into non-overlapping parts, the whole of the sum of the angle measures is the sum of the parts. (i.e. 1° wedge + 1° wedge + 1° wedge = 3° wedge or 3° angle.)<br>*Complementary and **supplementary angles based on pattern block work from prior day. It is more important that students understand that angles are additive than it is for them to know the vocabulary terms of complementary and supplementary angles at this point. |
| | **Mini-Lesson** | *In the last lesson we found a number of rules that will help us when thinking about lines and the angles that form when they intersect. Example: when lines are perpendicular the angle formed is 90°. Today we are going to figure out missing angle measures by using angle measures we already know.* Provide an example of a perpendicular line and ask students to estimate the angle measure – then use a protractor to prove it is 90° and notate it. Next ask what the adjacent angle is and how they know without using a protractor. Encourage varying answers, e.g. the straight line is 180°. If the angle we know is 90° then the angle we didn't measure must be 90° more. The straight line is half the circle, therefore ½ - ¼ is ¼ so its 90° or 360° ÷ 4 = 90°. Display another angle measure of 90°. Bisect the angle to create two 45° angles. Ask students again to estimate and then measure one of the angles. Then ask them to compute the other and justify their reasoning for the adjacent angle.<br>1. Tell students today they will try to find the missing angles on (three) different activity sheets. *Each sheet is progressively more challenging. The first asks that they estimate first then measure one angle, figure out the second without the protractor and then measure the second to see if they are correct with a sum of 90°.<br>2. Then provide them with Missing Angles sheet II & Missing Angles sheet III. They should try to figure out the missing angles with a sum of 180° without using a protractor and record how they found the angles. Tell them to be prepared to share their strategies during math congress. |
| | **Focus Questions for APS (1 min.)** | What is the missing angle?<br>What strategy did you use to find it? |
| (20-40 min.) | **Active Problem Solving** | As students work collaboratively, ask them to estimate the angle measure before they actually measure the angle. Encourage the use of mathematical terms like right, acute, and obtuse.<br>Once students are comfortable with accurately using the protractor to measure angles, encourage them to use their knowledge of degrees and angle relationships to help find solutions to today's problems. They can then check their work with a protractor for accuracy. |

| | | |
|---|---|---|
| | **Differentiation Suggestions** | For students who require support in seeing angle relationships, keep them grounded by using concrete models, wedges and the protractor. Focus on the amount of rotation using the protractor. Reinforce language like, 'How much of a turn is to the first angle measure?' and 'How much more of a turn do they need to make to get to 90° or 180°? Use concrete models to demonstrate the amount of turn and how much more is needed. Encourage additive thinking, e.g. 45° + 45° = 90°.<br>For students who need extension, encourage them to work on Missing Angles sheet III. You may also suggest that these students create some of their own missing angle sheets to challenge their peers. |
| | **Assessment Point** | Look for students who may confuse how to use and read the protractor, e.g. calling a 135° angle 45° or vs.<br>Who needs more practice with aligning the protractor on the vertex and ray (side of the angle)?<br>Who readily uses the protractor correctly and uses their understanding of angle relationships to determine the missing angle?<br>Note students who are using additive reasoning to find the missing angle measures. |
| (10 – 15 min.) | **Reconvene & Focus Q.** | Reconvene the class to the meeting area. Select a few angles from sheet I and ask students to convince their peers that they found the angle measures. Be explicit in recording student thinking using degrees and how students added the angle measures together. Then use some examples on missing angle sheet III to problem solve as whole group, as appropriate. |
| | **Journal Reflection Question** (2-5 min.) | What strategy did you use to find a missing angle? |
| | **Suggested Homework** (10-20 min.) | Your friend drives her car straight for 4 miles and turns right 90°. She continues to drive straight for 8 miles and turns right 90° again. She proceeds for 4 more miles straight and turns right, perpendicular to the direction she was just driving and continues driving straight for another 8 miles. What shapes did she outline with her drive?<br>Optional: Create your own driving story to outline another shape. |
| | **Notes for Next Lesson** | |

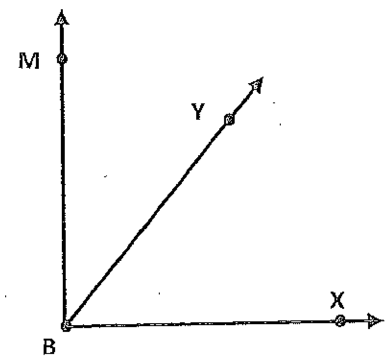
*Complementary angles when two angles add up to 90°

**Supplementary angles when two angles add up to 180°

**Fairfield Gr 4 U6 L11 S1**
**Complementary angles**

**Missing Angles sheet I**

1.
Write an estimate for the angle measure.
Measure one angle.
What is the complementary angle measure and how did you find it?



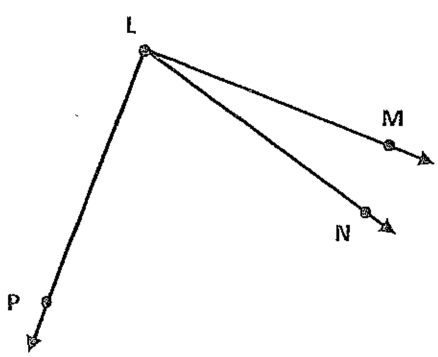

Estimate of ∠YBX ________°

∠YBX ________°

∠MBY ________°

How did you find ∠MBY?

2.



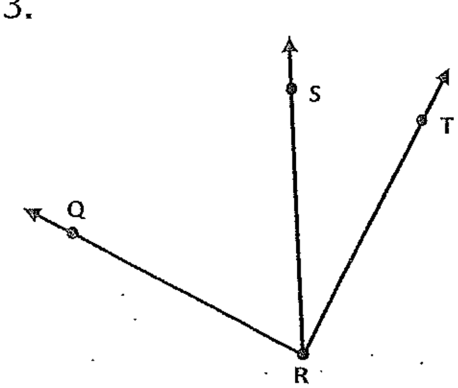

Estimate of ∠MLN ________°

∠MLN ________°

∠NLP ________°

How did you find ∠NLP?

3.



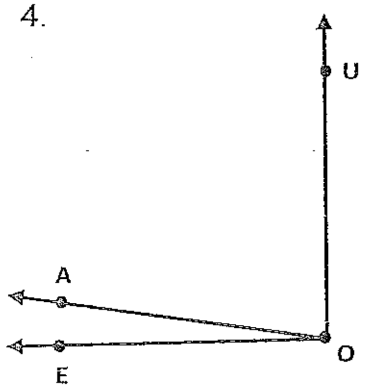

Estimate of ∠QRS ________°

∠QRS ________°

∠SRT ________°

How did you find ∠SRT?

4.



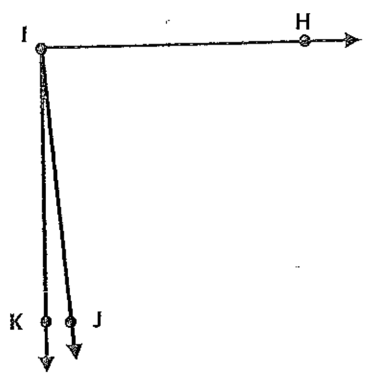

Estimate of ∠UOA ________°

∠UOA ________°

∠AOE ________°

How did you find ∠AOE?

5.



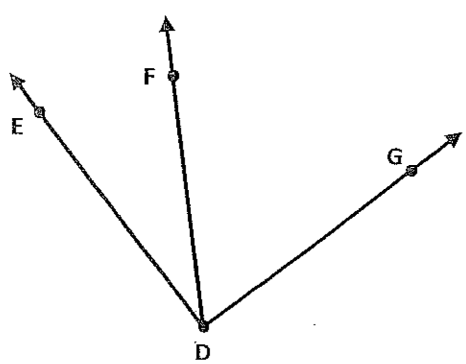

Estimate of ∠HIJ ______°

∠HIJ ______°

∠JIK ______°

How did you find ∠JIK?

6.




Estimate of ∠EDF ________°

∠EDF ________°

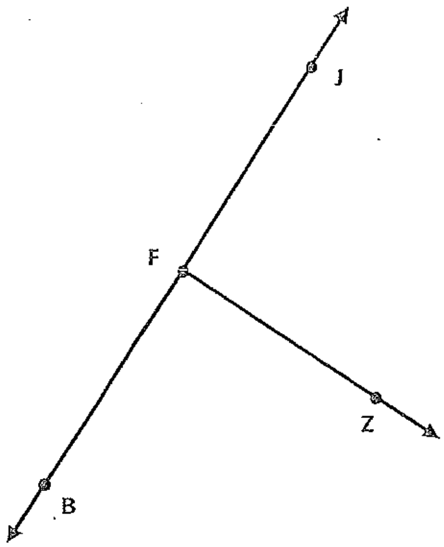
∠FDG ________°

How did you find ∠FDG?

**Fairfield GR 4 L11**
**Supplementary angles**

**Missing Angles sheet II**

1.



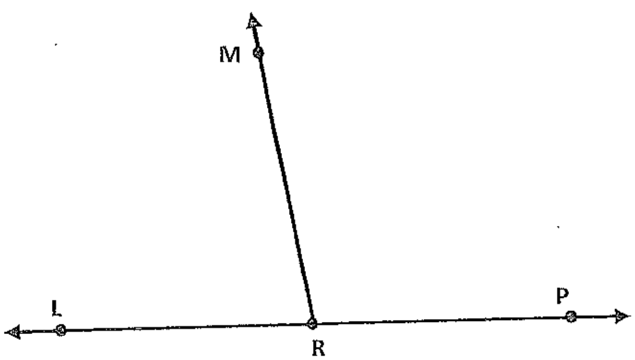

Estimate of ∠JFZ ________°

∠JFZ _____°

∠ZFB ________°

How did you find ∠ZFB?

2.



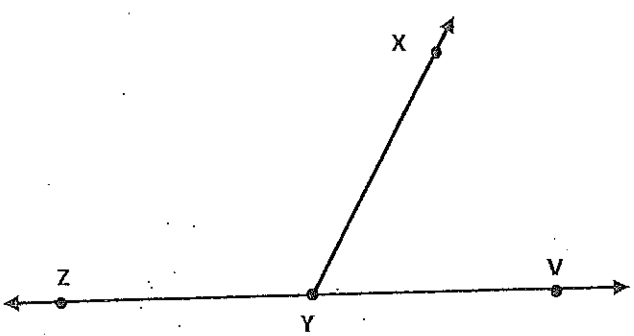

Estimate of ∠MRP ________°

∠MRP ________°

∠MRL ________°

How did you find ∠MRL?

3.



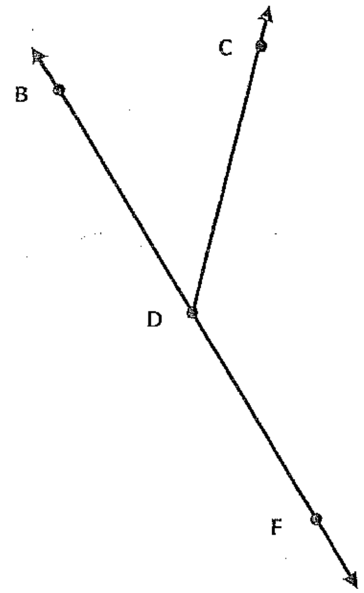

Estimate of ∠XYV ________°

∠XYV ________°

∠XYZ ________°

How did you find ∠XYZ?

4.



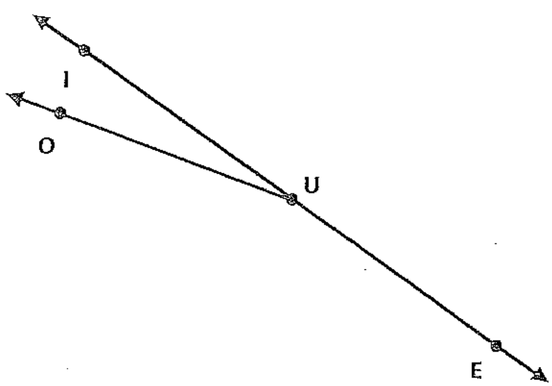

Estimate of ∠CDB __________°

∠CDB __________°

∠CDF __________°

How did you find ∠CDF?

5.



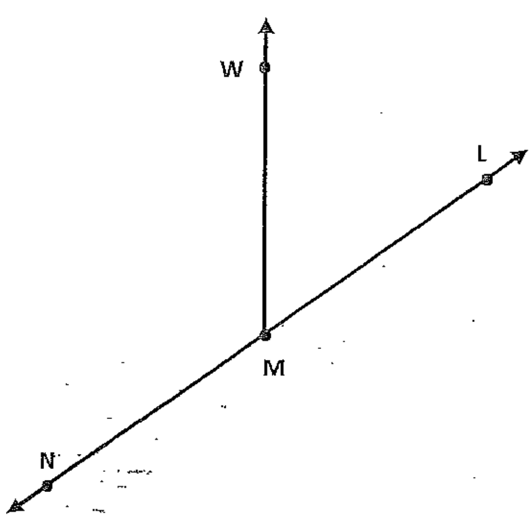

Estimate of ∠IUO __________°

∠IUO __________°

∠OUE __________°

How do you did you find ∠OUE?

6.




Estimate of ∠LWM __________°

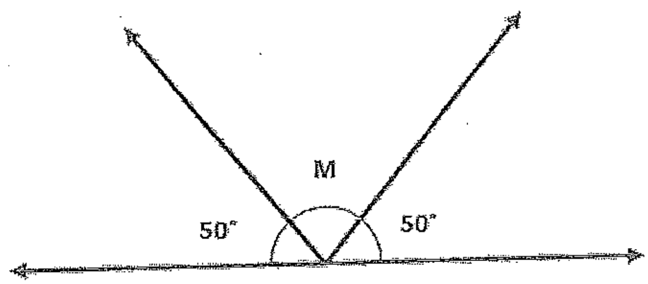
∠LWM __________°

∠WMN __________°

How did you find ∠WMN?

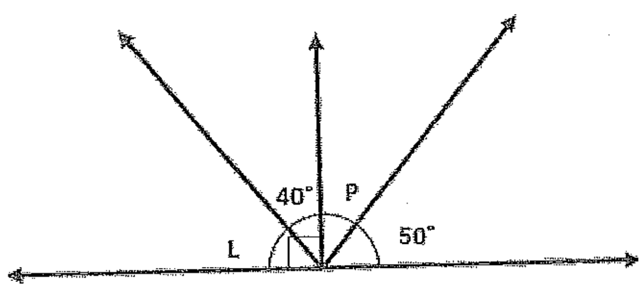
**Fairfield GR 4 L11** **Missing Angles sheet III**

1.



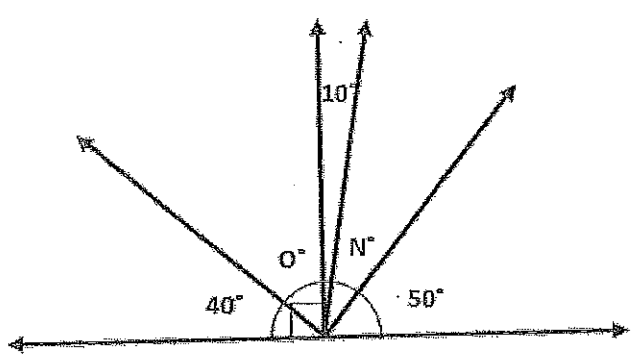

∠M ________°

2.



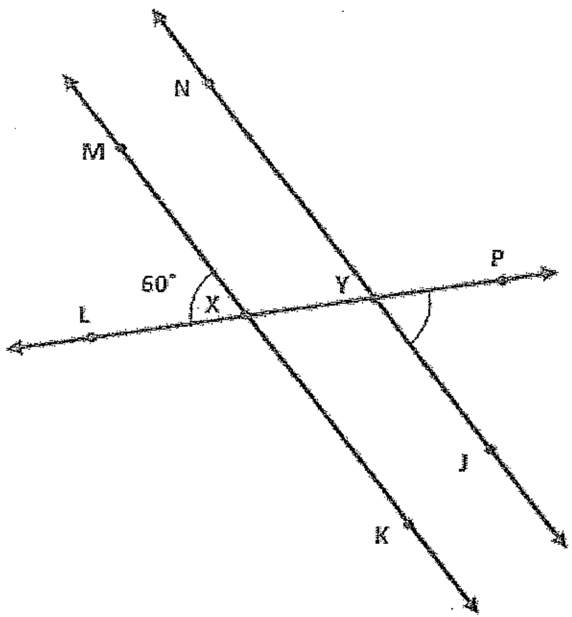

∠L ________°
∠P ________°

3.



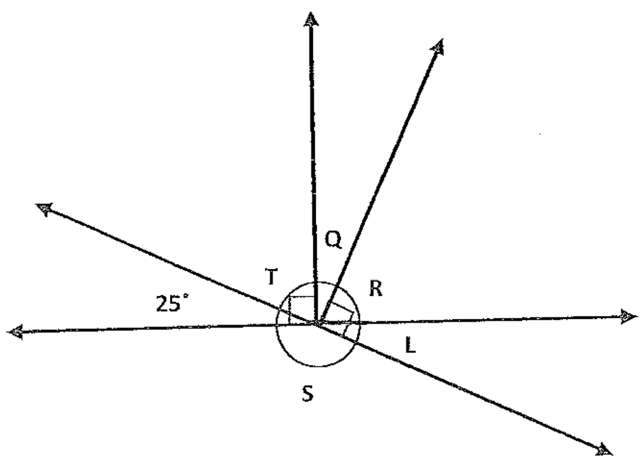

∠N ________°
∠O ________°

4. Lines $\overleftrightarrow{NJ}$ and $\overleftrightarrow{MK}$ are parallel.



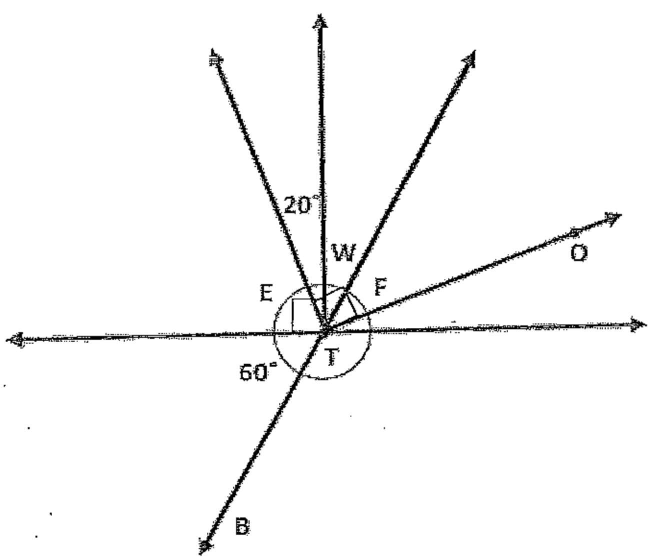

∠MXL ______60°
∠JYP ______°
∠KXL ______°
∠NYL ______°
∠NYP ______°
∠MXK ______°

5.




∠Q ______°
∠R ______°
∠S ______°
∠T ______°
∠L ______°

6.




∠BTO ______°
∠E ______°
∠W ______°
∠F ______°

| | | |
|---|---|---|
| | **Materials:** "A Ball Hit the Wall" sheets, "A Ball Hit the Wall" Recording sheet, tape, a ball for each group of students (any one of the following: ping pong, super ball, marble, recess ball, golf ball, foam ball), Books and ruler for a ramp, white paper, and protractors | |
| | **Fluency Work (1-5 min.)** | N/A today. Time will be needed for this investigation. |
| (5-10 min.) | **Teaching Point** | The purpose of today's lesson is to provide students with an opportunity to practice measuring angles in a context using a protractor. |
| | **Mini-Lesson** | In the meeting area, share with students that they will doing an investigation in which they need to measure and find the angles of a ball bouncing off the wall. (This may also be done on table-tops if they are level.) They will be working in teams of three where one will be rolling the ball, and the other two will be recording the point where the ball starts, the point where the ball bounces off the wall (or book), and the point where the ball ends up. *See A Ball Hit the Wall sheet for directions. It will be important to model your expectations and how to measure and record their findings. It is not necessary to have the ball approach the wall at the same angle every trial, but it is very important to try to be as accurate as possible when identifying the approaching and reflecting angles. When they are clear about the task, they should gather materials and conduct at least three trials recording their findings each time. |
| | **Focus Questions for APS (1 min.)** | If a ball hits the wall at one angle, at what angle does the ball bounce off the wall? |
| (20-40 min.) | **Active Problem Solving** | Provide students with the Ball Hits the Wall directions, recording sheet and materials needed. Monitor how accurately students are marking the approaching rays and reflecting rays. Encourage them to be as accurate as possible in measuring the angles – even careful measures will only be approximations. It is more important that students have the opportunity to appropriately measure angles and compare the angle measures they make than it is to do more than three trials. |
| | **Differentiation Suggestions** | For students who may have difficulty identifying the starting point of the approaching ray, provide them with a 'books and ruler (with a channel) ramp'. Align the ramp at a known angle by using a meter stick at a predetermined angle as a guide for the ramp. Then have them focus on where the ball hits the wall and where it bounces to.<br>This may also be done as a whole class activity with one set-up. Ask students to make predictions before each trial and then encourage them to Turn &Talk about their findings before they record them. |
| | **Assessment Point** | Note how students are measuring the angles. Do they line up the protractor properly? Are they using the correct reading of the angles? Are they able to determine which angles are acute and which are obtuse? What are they using as an initial ray? Are they able to determine the approaching ray and the reflected ray? |
| (10 – 15 min.) | **Reconvene & Focus Q.** | Reconvene the class and compare findings. What noticings do students make about the approaching angle and the reflected angle? *(they should be the same, also complementary angles should add up to 90°)* Can they make a general statement about the angle of an approaching angle to its reflected angle? Do they think it would be true if they changed the ball |

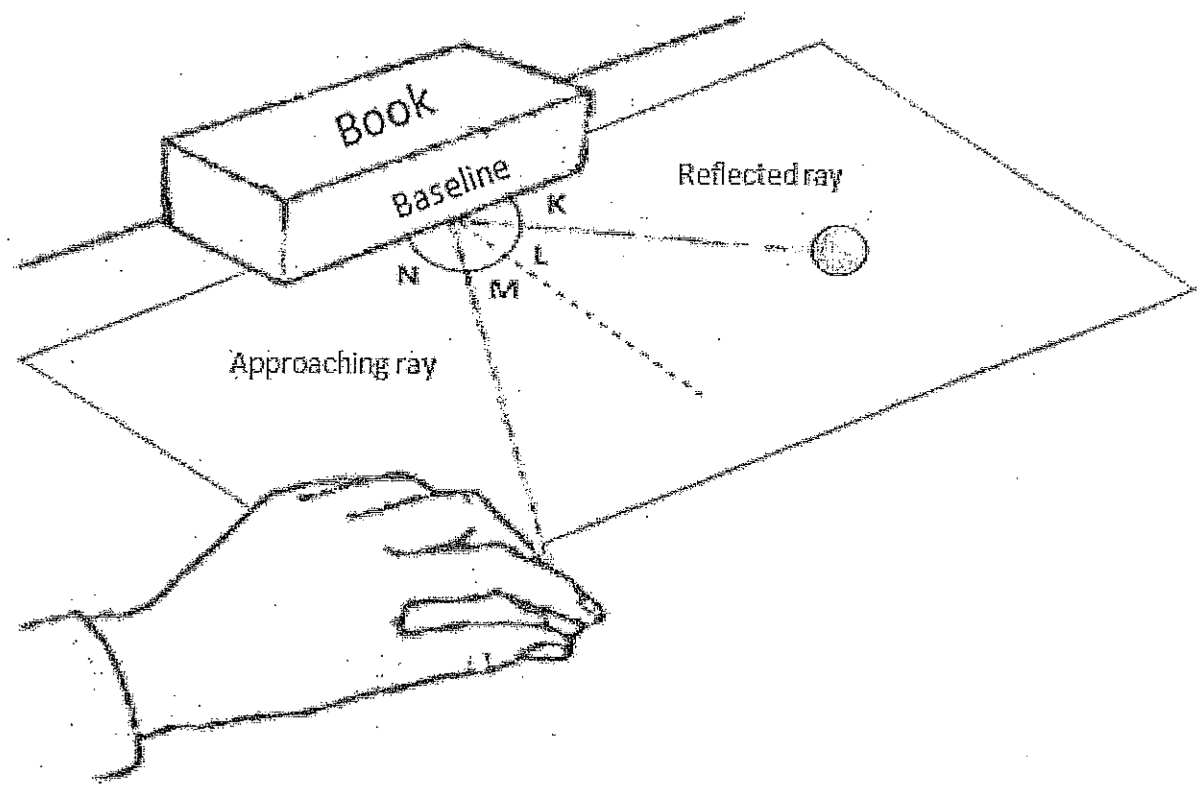
| | | |
|---|---|---|
| | | used or the surface it bounced off? Share strategies on how they actually used the protractor to measure the angle measures. If students measure the same angle and arrive at different angle measures, discuss why this might happen and what they need to remember when finding angle measures. |
| | **Journal Reflection Question** (2-5 min.) | What patterns did you notice when about the angle the ball approached the wall and the angle it bounced off the wall? |
| | **Suggested Homework** (10-20 min.) | 1. Define a square.<br>2. Define a rectangle.<br>3. How are they the same and how are they different? |
| | **Notes for Next Lesson** | |

## Materials

- *Book or hard object (the wall)*
- *Ping pong ball, rubber ball, marble,* or recess ball
- *Paper, white* 8 ½ x 11 or chart paper, *(soft carbon paper could also be used)*
- *Tape*

Directions:

1. Tape the paper against the book or wall
2. Roll the ball at the book (or wall).
3. Mark on the paper where the ball enters the paper. (Approaching ray)
4. Mark on the paper where the ball bounces off the book (wall).
5. Mark on the paper where the ball exits the paper. (Reflected ray)
6. Connect the marks for the approaching ray.
7. Connect the marks for the reflected ray.
8. Draw a line segment that is perpendicular to the book (wall) at the point where the ball hit.
9. Measure the approaching angle from both the wall and from the perpendicular line segment with your protractor.
10. Measure and compare the reflected angle.
11. Repeat this process for three trials and note any observations.



Name: ______________________________

Name: ______________________________

Name: ______________________________

| | Approaching Angle from the baseline | Reflected Angle from the baseline | Approaching angle from the perpendicular | Reflected angle from the perpendicular |
|---|---|---|---|---|
| Trial 1: | | | | |
| Trial 2: | | | | |
| Trial 3: | | | | |
| Trial 4: | | | | |
| Trial 5: | | | | |
| Trial 6: | | | | |

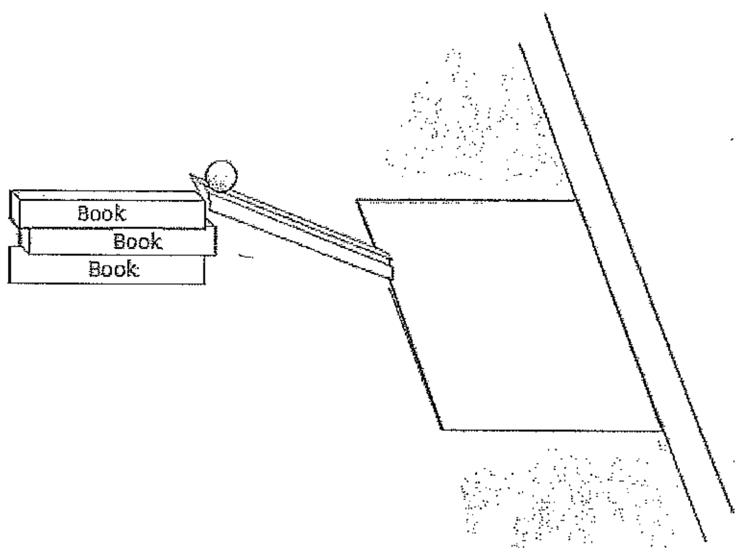
Compare the angle measures. What patterns do you notice?

Variation on this investigation:

- (Version 2)

1. Do the same task as outlined in the lesson but use chart paper on the floor against the wall and use a playground ball. Or,

2. Do the same task as outlined in the lesson above but use a ruler ramp to start the ball. This will provide a more accurate starting point for the approaching ray.




- (Version 3) Use rays of light:
  1. Make a slit in a box (shoe box)
  2. Place a flashlight in the box and direct it at a mirror
  3. Use modeling clay to support a mirror on the desk (or symmetry mirror)
  4. Place paper in front of the mirror
  5. Direct the box (light) at an angle across the paper toward the center of the mirror.
  6. Trace and label the approaching ray and the reflected ray
  7. Use a protractor to measure and compare the angles.

## Fairfield Public Schools Balanced Math Instructional Model



**Materials:** TSCM Vol.2 BLM 29 Assorted triangles, scissors, recording sheets or math journals.

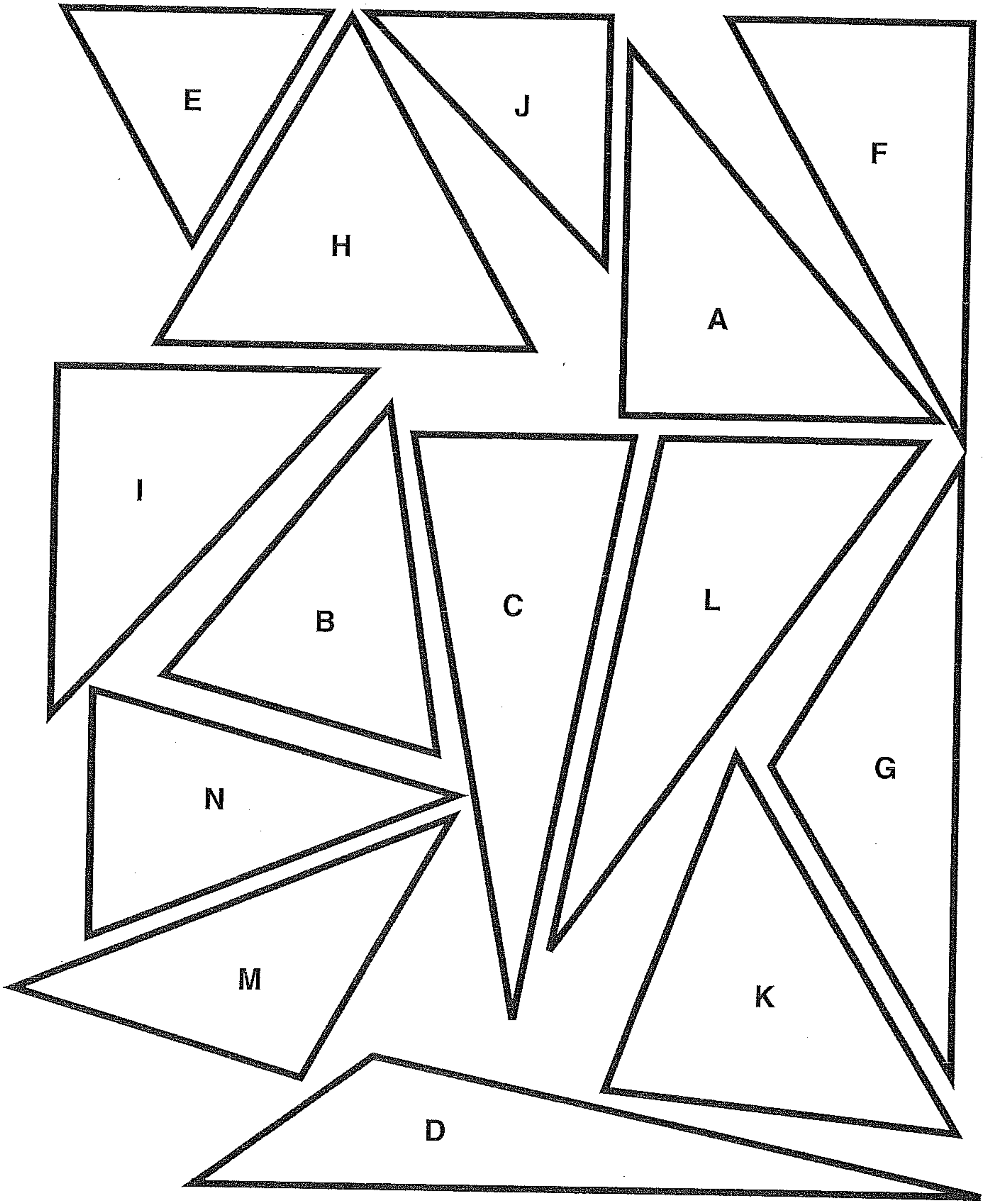
| | | |
|---|---|---|
| | **Fluency Work (1-5 min.)** | Create a ratio table and ask; "How many sides does a triangles have?" Draw (or ask students to draw) different triangles as models. Then ask how many side do 2 triangles have? 3 triangles, 5 triangles, 10, 12, 15, 25, 100, 110, 125? –record on the ratio table. Ask students if they can generate a rule to determine the number of sides for any amount of triangles. Three times the number of triangles = number of sides, or 3T=S or S÷3=T |
| (5-10 min.) | **Teaching Point** | Sort triangles and classify triangles: Defining types of triangles: equilateral triangle, isosceles triangle, scalene triangle |
| | **Mini-Lesson** | In the previous lesson we took a closer look at the sum of interior angles. Tell the students that today they are going to do a triangle sort. Ask students to define a triangle. Provide them time to T&T before sharing out. (a polygon or closed figure with exactly three sides and three angles) Share with them a few visual examples about how triangles can be different. Have student pairs cut out the triangles from the BLM 29 sheet. Their task is to sort the entire collection of triangles into three groups so that no triangle belongs to two groups. When complete, they need to write a description for each of the groups and be prepared to share their findings in math congress. Once done, they should find a second criterion for creating three different groupings and record how each group is different. Tell them they will have about twenty minutes to work with their partner and then the class will reconvene for a math congress. *Do not reveal the names of the types of triangles at this point. After triangles have been compared and sorted during the congress, then you can offer the "book" definition.* |
| | **Focus Questions for APS** | What are the different ways you can sort triangles into three different groups so that no triangle belongs to two groups? |
| (20-40 min.) | **Active Problem Solving** | Provide pairs of students with a copy of the BLM 29 Assorted Triangles sheet. As you confer with students notice their entry point into the problem. Ask probing questions about how they are grouping triangles and how they are convinced that a triangle does not fit into more than one group. Are they referencing what they know about angle measures and applying it to the angles of triangles, acute, right, and obtuse. Student will not have a gallery walk today before the math congress. Instead ask pairs to share with other pairs to compare their grouping strategies and encourage them to add anything to their descriptions that they may have left out or didn't notice. |
| | **Differentiation Suggestions** | For students who need support have a set of triangles pre-cut to physically manipulate to compare. Students may need support to focus their attention on attributes like angle sizes or congruent sides. Student may use the corner of an index card to determine right angles or to compare lengths of sides. For students who need extension ask them to find triangles that fit in two groups using a Venn diagram and record the attributes that are different & attributes that are in common, *e.g. right triangles and isosceles triangles may have different length sides but both can have a 90°* |

| | | |
|---|---|---|
| | **Assessment Point** | As you confer with students notice their entry point into the problem. Are they grouping triangles because they appear to be alike? Are they grouping them by their attributes? Are they going beyond this and noticing relationships and properties? Also note who is using math terms such as acute, obtuse, right angles, congruent, similar, perpendicular, and vertex. |
| (10 – 15 min.) | **Reconvene & Focus Q.** | Reconvene the class to the meeting area. Ask student groups to lead the discussion and begin to make a class record of the attributes students used to group triangles. Ask students to compare triangles by attributes (length of sides, comparisons of sides, size of angles, types of angles) Make a point of organizing the recorded list by types of triangles. After the lists are well defined offer the definitions for each type or category of triangle (below). Then pose the question can a triangle belong under two different classifications and if so, how? Provide think-time and ask students to turn and talk before sharing their reasoning. Encourage the use of math terms during discussion: perpendicular lines, angle degrees, right angles, congruent sides... Record triangles that fit under more than one classification with an explanation of why. Compare the classifications by asking how they are the same and how are they different?<br>*see<br>*TSCM V.2 p. 221*<br>*<u>Classified by sides</u>*<br>*Equilateral -All sides are congruent*<br>*Isosceles -At least two sides are congruent*<br>*Scalene -No two sides are congruent*<br><br>*<u>Classified by angles</u>*<br>*Right -Has a right angle*<br>*Acute -All angles are smaller than a right angle*<br>*Obtuse -One angle is larger than a right angle* |
| | **Journal Reflection Question** (2-5 min.) | What is an equilateral triangle?<br>What is an isosceles triangle?<br>What is a scalene triangle? |
| | **Suggested Homework** (10-20 min.) | SF p. 446, 1-5, 12-16 |
| | **Notes for Next Lesson** | |

**TABLE 8.1**
Categories of Two-Dimensional Shapes

| *Shape* | *Description* |
|---|---|
| **Simple Closed Curves** | |
| Concave, convex | An intuitive definition of *concave* might be "having a dent in it." If a simple closed curve is not concave, it is *convex.* A more precise definition of *concave* may be interesting to explore with older students. |
| Symmetrical, nonsymmetrical | Shapes may have one or more lines of symmetry and may or may not have rotational symmetry. These concepts will require more detailed investigation. |
| Polygons | Simple closed curves with all straight sides. |
| Concave, convex | |
| Symmetrical, nonsymmetrical | |
| Regular | All sides and all angles are congruent. |
| **Triangles** | |
| Triangles | Polygons with exactly three sides. |
| *Classified by sides* | |
| Equilateral | All sides are congruent. |
| Isosceles | At least two sides are congruent. |
| Scalene | No two sides are congruent. |
| *Classified by angles* | |
| Right | Has a right angle. |
| Acute | All angles are smaller than a right angle. |
| Obtuse | One angle is larger than a right angle. |
| **Convex Quadrilaterals** | |
| Convex quadrilaterals | Convex polygons with exactly four sides. |
| Kite | Two opposing pairs of congruent adjacent sides. |
| Trapezoid | At least one pair of parallel sides. |
| Isosceles trapezoid | A pair of opposite sides is congruent. |
| Parallelogram | Two pairs of parallel sides. |
| Rectangle | Parallelogram with a right angle. |
| Rhombus | Parallelogram with all sides congruent. |
| Square | Parallelogram with a right angle and all sides congruent. |

Many textbooks define cylinders strictly as circular cylinders. These books do not have special names for other cylinders. Under that definition, the prism is not a special case of a cylinder. This points to the fact that definitions are conventions, and not all conventions are universally agreed upon. If you look at the development of the volume formulas in Chapter 9, you will see that the more inclusive definition of cylinders and cones given here allows one formula for any type of cylinder—hence, prisms—with a similar statement that is true for cones and pyramids.


E
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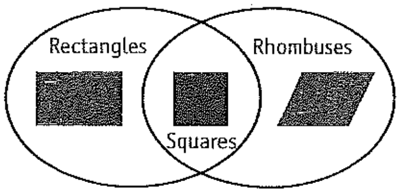
## CHECK

**Error Intervention**

**If** students have difficulty understanding how a particular triangle or quadrilateral can belong to more than one set,

**then** show and discuss Venn diagrams such as the one below.




*(Also see* Reteaching, *p. 444B.)*

## 3 Practice

**Exercise 23** Remind students to use math terms accurately. Answers should include the general description of a term (A rectangle has four right angles) and how a part of the flag satisfies that description (The rectangles on Chile's flag have 4 right angles).

**Reading Assist Draw Conclusions Can a triangle be classified as both acute and obtuse?** *(No; An acute triangle must have three acute angles; an obtuse triangle has one obtuse angle.)* **Can a triangle be classified as both equilateral and acute?** *(Yes; When all sides are the same length, each angle is an acute angle.)*

**Leveled Practice**

Below Level Ex. 12–21, 23, 25–27

On Level Ex. 12–21, 23, 25–27

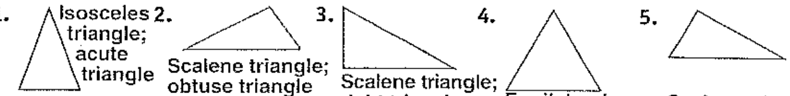
Above Level Ex. 12–16 even, 17–27



## CHECK

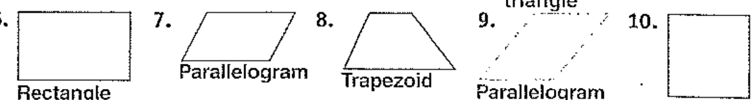
*For another example, see Set 8-4 on p. 491.*

Classify each triangle by its sides and then by its angles.



1. Isosceles triangle; acute triangle
2. Scalene triangle; obtuse triangle
3. Scalene triangle; right triangle
4. Equilateral triangle; acute triangle
5. Scalene triangle; acute triangle

Write the name of each quadrilateral.



6. Rectangle
7. Parallelogram
8. Trapezoid
9. Parallelogram
10. Square

11. **Reasoning** Can a quadrilateral be both a square and a rectangle? Explain. **Yes, all squares are a specific type of rectangle.**

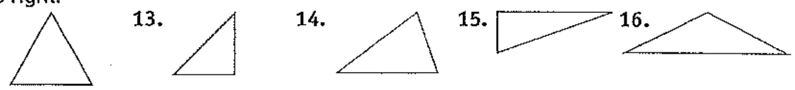
## PRACTICE

*For more practice, see Set 8-4 on p. 494.*

### A Skills and Understanding

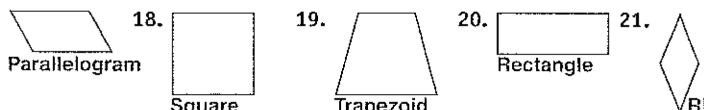
Classify each triangle by its sides and then by its angles. **See right.**

**12. Equilateral triangle; acute triangle**
**13. Isosceles triangle; right triangle**
**14. Scalene triangle; acute triangle**
**15. Scalene triangle; right triangle**
**16. Isosceles triangle; obtuse triangle**



12. 13. 14. 15. 16.

Write the name of each quadrilateral.



17. Parallelogram
18. Square
19. Trapezoid
20. Rectangle
21. Rhombus

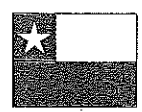
22. **Reasoning** Can a quadrilateral be both a parallelogram and a rhombus? Explain. **Yes, a rhombus is a specific type of parallelogram.**

### B Reasoning and Problem Solving

**Math and Social Studies**

Many flags are designed with a variety of geometric figures.

*The Czech Republic is located in central Europe.*



*Chile is a country in southwest South America.*

23. **Writing in Math** Use mathematical terms from this lesson to describe the shapes that are used in each flag. **Czech Republic's flag has 2 trapezoids and one triangle. Chile's flag has 2 rectangles, one square, and a 10-sided star.**

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## Test-Taking Practice

**Test-Taking Practice, Item 1, p. 447**
There are four parts to the question. Each term with its corresponding explanation is worth one point, and having everything correct is also worth one point for four points total. Remind students that they should answer as much as they can since partial credit is given for this type of problem.

Discuss the sample responses shown and compare them to papers produced by your students.

**4-point answer** All terms and exp tions are correct; shows complete understanding of quadrilaterals.

The shape is a quadrilateral because it has 4 sides and 4 angles. It is a rectangle because it has 4 right angles. is a square because it has 4 right angles and 4 equal sides

## *Fairfield Public Schools*

## *District-Wide Mathematics Benchmark Assessments*

| Grade | Assessment | Areas Assessed | Administration Dates |
|---|---|---|---|
| Preschool<br>(ages 3 & 4) | Counting<br><br>Cardinality | Oral Counting<br><br>Tells "how many" | January, May<br><br>January , May |
| Kindergarten | Test of Early Numeracy | Oral Counting<br>Number Identification<br>Quantity discrimination<br>Missing Number | October, January, May |
| Kindergarten | End of Unit Assessments | Skills assessed related to unit taught | At the end of each unit (approximately every 4-6 weeks) |
| Grade 1 | Test of Early Numeracy<br><br>Math Computation Test | Oral Counting<br>Number Identification<br>Quantity discrimination<br>Missing Number<br><br>Basic addition and subtraction facts | October, January, May |
| Grade 1 | End of Unit Assessments | Skills assessed related to unit taught | At the end of each unit (approximately every 4-6 weeks) |
| Grade 2 | Math Computation Test<br><br>Concepts and Application Test | Basic addition and subtraction facts<br><br>Problem solving and computation test | October, January, May |
| Grade 2 | End of Unit Assessments | Skills assessed related to unit taught | At the end of each unit (approximately every 4-6 weeks) |

**Fairfield Public Schools**
**Connecticut Mastery Test Results**
**Grade 3**
**2007-2012**
**Percent at / above Goal**

| | Reading | | | | | | Writing | | | | | | Math | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 2007 | 2008 | 2009 | 2010 | 2011 | 2012 | 2007 | 2008 | 2009 | 2010 | 2011 | 2012 | 2007 | 2008 | 2009 | 2010 | 2011 | 2012 |
| Burr | 67.4 | 75.7 | 87.0 | 86.0 | 90.9 | 81.2 | 76.3 | 79.7 | 87.3 | 82.8 | 83.9 | 85.2 | 76.3 | 83.8 | 79.7 | 91.2 | 87.3 | 90.7 |
| Dwight | 86.5 | 93.5 | 76.4 | 82.4 | 88.1 | 92.9 | 92.3 | 88.5 | 76.8 | 77.8 | 83.6 | 92.9 | 88.5 | 95.2 | 83.6 | 88.2 | 91.5 | 97.6 |
| Holland Hill | 59.3 | 79.7 | 76.9 | 83.6 | 79.3 | 78.0 | 78.0 | 84.7 | 77.8 | 93.4 | 86.9 | 85.0 | 79.7 | 84.7 | 92.5 | 85.2 | 80.0 | 84.7 |
| Jennings | 69.0 | 71.7 | 68.7 | 71.9 | 78.9 | 66.0 | 72.4 | 71.7 | 77.6 | 60.9 | 71.2 | 86.0 | 74.1 | 88.7 | 82.1 | 78.1 | 86.0 | 98.0 |
| McKinley | 46.3 | 48.6 | 58.7 | 52.9 | 68.3 | 41.2 | 67.2 | 54.2 | 66.7 | 50.0 | 65.0 | 64.2 | 61.2 | 55.6 | 71.0 | 58.6 | 75.0 | 69.1 |
| Mill Hill | 80.9 | 79.7 | 73.9 | 80.8 | 74.3 | 70.5 | 88.1 | 74.7 | 84.1 | 74.1 | 77.0 | 83.3 | 87.0 | 78.5 | 85.2 | 84.8 | 87.8 | 85.9 |
| N. Stratfield | 68.0 | 71.4 | 73.4 | 71.4 | 73.8 | 68.2 | 74.7 | 65.1 | 72.2 | 70.3 | 83.8 | 77.9 | 84.2 | 77.4 | 86.1 | 86.8 | 86.3 | 84.7 |
| Osborn Hill | 80.7 | 78.7 | 66.3 | 78.5 | 71.4 | 80.9 | 86.4 | 84.3 | 80.2 | 70.2 | 75.0 | 70.3 | 79.5 | 80.9 | 72.1 | 73.1 | 73.8 | 90.0 |
| Riverfield | 76.3 | 71.1 | 62.6 | 75.9 | 65.6 | 83.6 | 75.0 | 75.0 | 78.9 | 71.3 | 70.8 | 84.9 | 77.5 | 78.9 | 69.9 | 82.3 | 70.8 | 86.3 |
| Sherman | 70.1 | 86.7 | 71.7 | 76.5 | 83.3 | 76.7 | 80.6 | 80.0 | 73.8 | 76.7 | 85.9 | 74.2 | 64.7 | 81.1 | 85.0 | 80.0 | 78.2 | 76.7 |
| Stratfield | 59.8 | 75.3 | 89.5 | 88.9 | 70.6 | 85.3 | 68.8 | 79.2 | 80.2 | 78.2 | 74.3 | 78.5 | 74.2 | 79.5 | 88.5 | 88.2 | 91.2 | 86.4 |
| District Avg. | 69.2 | 75.7 | 73.0 | 76.2 | 76.4 | 75.0 | 77.6 | 76.0 | 78.2 | 72.1 | 78.1 | 79.5 | 76.7 | 79.9 | 80.8 | 80.5 | 82.3 | 85.9 |
| State Avg. | 52.3 | 52.1 | 54.6 | 57.1 | 58.4 | 59.2 | 60.8 | 63.5 | 62.6 | 58.3 | 61.1 | 62.7 | 59.4 | 60.2 | 63.0 | 62.6 | 63.3 | 66.8 |

These results do not reflect the newly adopted grades 3-5 math curriculum as approved by the Board of Education in April 2012.

**Fairfield Public Schools**
**Connecticut Mastery Test Results**
**Grade 4**
**2007-2011**
**Percent at / above Goal**

| | Reading | | | | | | Writing | | | | | | Math | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | **2007** | **2008** | **2009** | **2010** | **2011** | **2012** | **2007** | **2008** | **2009** | **2010** | **2011** | **2012** | **2007** | **2008** | **2009** | **2010** | **2011** | **2012** |
| **Burr** | 89.0 | 73.2 | 85.3 | 84.7 | 81.4 | **91.5** | 90.3 | 80.2 | 90.4 | 93.2 | 83.9 | **85.2** | 87.7 | 77.3 | 88.6 | 94.5 | 96.6 | **94.8** |
| **Dwight** | 85.7 | 87.5 | 90.2 | 92.9 | 85.2 | **91.2** | 87.5 | 87.5 | 85.2 | 84.5 | 90.9 | **93.0** | 87.5 | 91.1 | 91.8 | 92.9 | 92.6 | **93.0** |
| **Holland Hill** | 71.4 | 70.2 | 82.1 | 67.9 | 84.5 | **77.8** | 78.6 | 80.7 | 87.7 | 76.4 | 86.2 | **86.2** | 69.6 | 70.2 | 82.5 | 74.1 | 84.5 | **69.1** |
| **Jennings** | 70.4 | 68.3 | 76.0 | 82.4 | 78.5 | **81.4** | 68.5 | 76.3 | 81.1 | 83.8 | 76.9 | **72.6** | 74.1 | 68.3 | 82.0 | 79.4 | 78.5 | **81.7** |
| **McKinley** | 65.1 | 57.8 | 46.3 | 60.0 | 63.1 | **67.7** | 62.8 | 67.2 | 50.0 | 69.1 | 64.4 | **57.8** | 67.8 | 67.2 | 51.8 | 66.2 | 55.3 | **67.2** |
| **Mill Hill** | 70.1 | 89.0 | 80.3 | 79.1 | 79.2 | **77.6** | 88.1 | 91.8 | 83.3 | 83.7 | 74.0 | **84.2** | 89.6 | 78.1 | 80.5 | 84.9 | 81.1 | **82.9** |
| **N. Stratfield** | 71.4 | 76.5 | 73.4 | 81.8 | 73.7 | **73.1** | 77.6 | 77.5 | 81.5 | 80.5 | 77.9 | **85.9** | 87.0 | 86.4 | 83.3 | 92.2 | 82.1 | **92.3** |
| **Osborn Hill** | 75.0 | 84.3 | 81.5 | 76.4 | 81.1 | **83.7** | 88.8 | 79.8 | 84.3 | 80.0 | 85.9 | **81.6** | 85.0 | 87.6 | 87.8 | 85.2 | 78.9 | **87.2** |
| **Riverfield** | 78.3 | 78.8 | 81.8 | 71.3 | 83.8 | **69.2** | 81.2 | 74.7 | 85.9 | 84.0 | 85.2 | **78.8** | 84.1 | 78.5 | 87.0 | 77.7 | 81.3 | **78.5** |
| **Sherman** | 73.8 | 78.4 | 91.4 | 86.9 | 90.9 | **82.1** | 92.3 | 82.4 | 81.7 | 84.1 | 96.2 | **93.6** | 90.8 | 70.3 | 78.5 | 87.1 | 93.5 | **85.9** |
| **Stratfield** | 91.7 | 76.5 | 79.5 | 85.4 | 91.3 | **83.6** | 89.6 | 80.6 | 82.1 | 87.2 | 86.0 | **75.7** | 89.6 | 79.6 | 82.1 | 86.6 | 91.6 | **86.6** |
| **District Avg.** | 75.7 | 76.6 | 78.5 | 78.7 | 80.6 | **79.8** | 81.7 | 79.8 | 80.4 | 82.5 | 81.9 | **81.5** | 82.6 | 78.0 | 80.7 | 83.7 | 82.3 | **83.9** |
| **State Avg.** | 57.0 | 56.0 | 60.7 | 60.0 | 62.5 | **64.1** | 65.1 | 62.9 | 64.2 | 64.2 | 65.5 | **65.3** | 62.3 | 60.5 | 63.8 | 67.2 | 67.3 | **68.2** |

These results do not reflect the newly adopted grades 3-5 math curriculum as approved by the Board of Education in April 2012.

**Fairfield Public Schools**
**Connecticut Mastery Test Results**
**Grade 5**
**2007-2011**
**Percent at / above Goal**

| | Reading | | | | | | Writing | | | | | | Math | | | | | | Science | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 2007 | 2008 | 2009 | 2010 | 2011 | 2012 | 2007 | 2008 | 2009 | 2010 | 2011 | 2012 | 2007 | 2008 | 2009 | 2010 | 2011 | 2012 | 2008 | 2009 | 2010 | 2011 | 2012 |
| Burr | 84.5 | 88.0 | 83.5 | 85.3 | 90.0 | 82.5 | 84.7 | 93.3 | 79.4 | 91.5 | 91.4 | 83.3 | 88.2 | 89.3 | 84.2 | 92.8 | 92.9 | 91.9 | 88.0 | 75.0 | 75.0 | 88.6 | 77.3 |
| Dwight | 84.3 | 80.4 | 85.5 | 86.4 | 89.3 | 88.7 | 90.2 | 94.6 | 90.9 | 93.2 | 89.7 | 81.5 | 92.2 | 91.1 | 94.5 | 94.9 | 91.1 | 88.7 | 80.4 | 78.2 | 89.8 | 81.0 | 85.2 |
| Holland Hill | 58.6 | 71.2 | 80.7 | 76.3 | 66.7 | 80.7 | 62.1 | 79.7 | 73.3 | 88.1 | 63.3 | 84.2 | 75.9 | 74.6 | 84.2 | 88.1 | 76.3 | 92.7 | 71.2 | 70.0 | 67.8 | 68.3 | 80.7 |
| Jennings | 79.6 | 73.2 | 77.2 | 75.5 | 73.9 | 84.1 | 83.3 | 85.5 | 82.0 | 78.8 | 82.6 | 87.3 | 85.2 | 71.4 | 77.2 | 85.7 | 82.6 | 85.9 | 60.0 | 70.5 | 63.5 | 76.8 | 84.4 |
| McKinley | 52.8 | 73.2 | 72.3 | 45.5 | 63.8 | 61.4 | 56.9 | 65.9 | 70.1 | 40.4 | 75.0 | 61.5 | 61.1 | 69.5 | 68.2 | 61.4 | 73.2 | 55.6 | 70.7 | 73.1 | 49.4 | 65.3 | 53.8 |
| Mill Hill | 92.3 | 81.7 | 84.5 | 84.5 | 81.0 | 87.5 | 92.3 | 85.7 | 94.4 | 89.2 | 89.3 | 88.2 | 94.9 | 87.1 | 90.1 | 88.0 | 89.3 | 86.5 | 70.4 | 83.1 | 77.3 | 82.1 | 77.6 |
| N. Stratfield | 80.8 | 82.4 | 82.3 | 74.0 | 79.2 | 75.5 | 67.1 | 74.3 | 80.5 | 69.2 | 77.9 | 84.9 | 83.6 | 86.5 | 90.0 | 85.7 | 94.8 | 85.1 | 75.7 | 72.0 | 70.5 | 79.2 | 73.4 |
| Osborn Hill | 81.6 | 83.8 | 87.2 | 82.9 | 84.1 | 87.1 | 78.9 | 85.0 | 90.8 | 88.1 | 75.6 | 86.3 | 84.2 | 83.8 | 93.1 | 90.4 | 87.4 | 90.3 | 86.3 | 90.8 | 88.1 | 87.8 | 84.2 |
| Riverfield | 79.7 | 74.3 | 87.3 | 80.0 | 74.0 | 90.8 | 78.4 | 75.7 | 85.4 | 78.8 | 81.0 | 88.2 | 84.9 | 78.6 | 86.4 | 95.0 | 89.0 | 90.9 | 68.6 | 84.1 | 82.5 | 81.0 | 91.0 |
| Sherman | 78.5 | 87.5 | 80.8 | 87.2 | 88.9 | 90.5 | 82.8 | 92.2 | 86.7 | 92.6 | 87.7 | 91.7 | 76.9 | 90.6 | 83.3 | 86.2 | 90.5 | 97.6 | 92.2 | 81.3 | 86.2 | 89.2 | 95.2 |
| Stratfield | 69.6 | 91.8 | 83.0 | 80.8 | 84.3 | 88.0 | 77.2 | 91.8 | 86.1 | 78.2 | 75.9 | 80.7 | 75.9 | 89.8 | 81.0 | 86.1 | 93.9 | 94.1 | 89.8 | 73.3 | 74.7 | 75.9 | 83.0 |
| District Avg. | 76.8 | 80.2 | 82.4 | 77.6 | 79.4 | 82.8 | 77.6 | 83.0 | 83.6 | 79.9 | 80.7 | 82.9 | 82.1 | 82.5 | 84.8 | 86.0 | 87.5 | 86.6 | 77.4 | 77.5 | 75.1 | 79.8 | 80.0 |
| State Avg. | 61.5 | 62.2 | 66.0 | 61.8 | 61.4 | 67.7 | 64.6 | 64.6 | 66.6 | 68.2 | 66.8 | 68.1 | 66.0 | 66.2 | 69.0 | 72.6 | 72.7 | 71.8 | 55.2 | 58.3 | 59.7 | 60.2 | 64.1 |

These results do not reflect the newly adopted grades 3-5 math curriculum as approved by the Board of Education in April 2012.

## *Basic Facts*

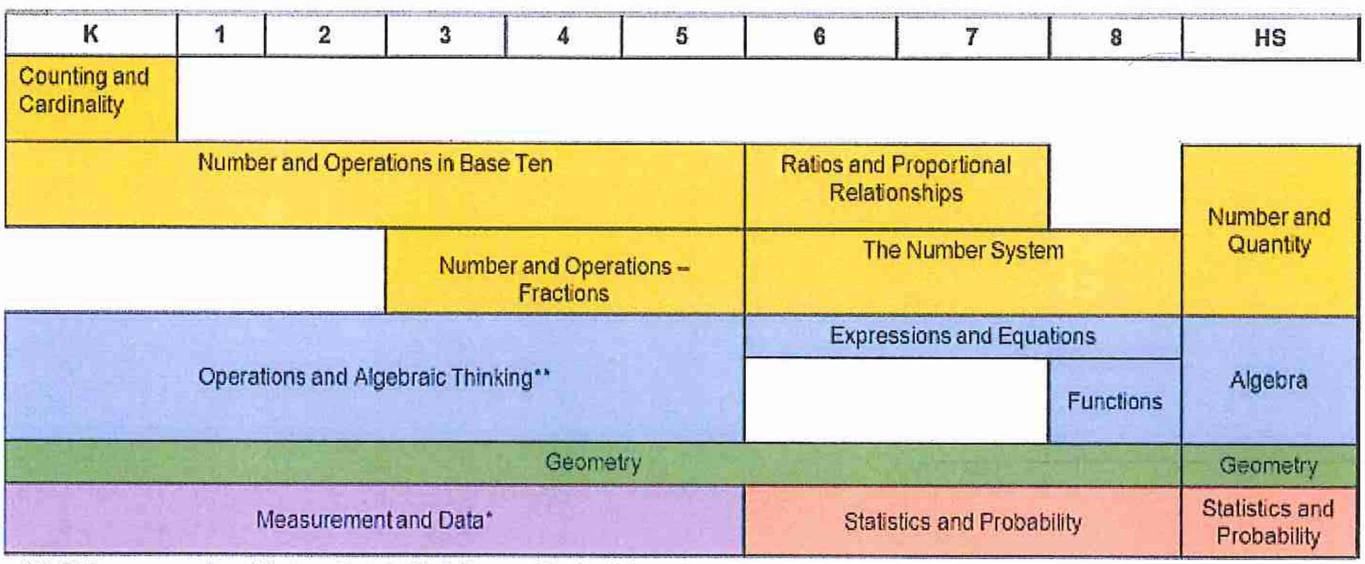
The fluency test of basic facts is a 1 minute test of basic fact recall administered to all students in grade five. It measures how many basic fact problems students are able to complete accurately in one minute with all four operations. The data gives the percentage of students who complete up to twenty problems in one minute, between 20 and 30 problems in one minute, and above 30 problems in one minute.

| Exiting Grade 5 Fluency Test Results between 2006 to 2009 and 2010 to 2012 | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | Addition | | Subtraction | | Multiplication | | Division | |
| Number of problems completed in a minute | 2006 to 2009 | 2010 to 2012 | 2006 to 2009 | 2010 to 2012 | 2006 to 2009 | 2010 to 2012 | 2006 to 2009 | 2010 to 2012 |
| 0-20 | 17% | 11% | 37% | 24% | 18% | 18% | 44% | 31% |
| 21-30 | 36% | 29% | 40% | 37% | 39% | 39% | 31% | 32% |
| 30-up | 47% | 60% | 23% | 39% | 43% | 43% | 25% | 37% |

*PK-12 Continuum of Mathematics Domains and Standards*

The diagram below demonstrates the PK-12 focus and coherence of mathematics domains. Attached is a sample of the progression of one domain from preschool to grade 6. It demonstrates the place value standards across eight grade levels.

# Domains across the grade levels




* K-5 Measurement and Data splits into Statistics and Probability and Geometry in Grade 6

**Operations and Algebraic Thinking is foundation for Grade 6 Expressions and Equations and The Number System

# Place Value Coherence among the grades

## PK Counting and Cardinality

PK.NBT.1 Compose and decompose numbers (ex. Twelve is ten and two)

PK.NBT.2 Recognize numbers are contained in other numbers (hierarchical inclusion: four is contained in the number five and five is one more than four)

## Grade K Work with numbers 11-19 to gain foundations for place value

K.NBT.1 Compose and decompose numbers from 11 to 19 into ten ones and some further ones, e.g., by using objects or drawings, and record each composition or decomposition by a drawing or equation (such as 18 = 10 + 8); understand that these numbers are composed of ten ones and one, two, three, four, five, six, seven, eight, or nine ones.

## Grade 1 Understand place value.

1.NBT.B.2 Understand that the two digits of a two-digit number represent amounts of tens and ones. Understand the following as special cases:

- 1.NBT.B.2a 10 can be thought of as a bundle of ten ones — called a "ten."
- 1.NBT.B.2b The numbers from 11 to 19 are composed of a ten and one, two, three, four, five, six, seven, eight, or nine ones.
- 1.NBT.B.2c The numbers 10, 20, 30, 40, 50, 60, 70, 80, 90 refer to one, two, three, four, five, six, seven, eight, or nine tens (and 0 ones).

## Grade 2 Understand place value.

2.NBT.1 Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones; e.g., 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases:

- 2.NBT.1a 100 can be thought of as a bundle of ten tens — called a "hundred."
- 2.NBT.1b The numbers 100, 200, 300, 400, 500, 600, 700, 800, 900 refer to one, two, three, four, five, six, seven, eight, or nine hundreds (and 0 tens and 0 ones).
- 2.NBT.B.5 Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.
- 2.NBT.B.6 Add up to four two-digit numbers using strategies based on place value and properties of operations.

## Grade 3 Use place value understanding and properties of operations to perform multi-digit arithmetic

- 3.NBT.1 Use place value understanding to round whole numbers to the nearest 10 or 100.
- 3.NBT.2 Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction.

## Grade 4 Generalize place value understanding for multi-digit whole numbers.

- 4.NBT.A.1 Recognize that in a multi-digit whole number, a digit in one place represents ten times what it represents in the place to its right. *For example, recognize that $700 \div 70 = 10$ by applying concepts of place value and division.*
- 4.NBT.A.2 Read and write multi-digit whole numbers using base-ten numerals, number names, and expanded form. Compare two multi-digit numbers based on meanings of the digits in each place, using >, =, and < symbols to record the results of comparisons.
- 4.NBT.A.3 Use place value understanding to round multi-digit whole numbers to any place.

## Grade 5 Understand the place value system.

- 5.NBT.1 Recognize that in a multi-digit number, a digit in one place represents 10 times as much as it represents in the place to its right and 1/10 of what it represents in the place to its left.
- 5.NBT.2 Explain patterns in the number of zeros of the product when multiplying a number by powers of 10, and explain patterns in the placement of the decimal point when a decimal is multiplied or divided by a power of 10. Use whole-number exponents to denote powers of 10.

## Grade 6 Apply and extend previous understandings of numbers to the system of rational numbers.

- 6.NS.C.5 Understand that positive and negative numbers are used together to describe quantities having opposite directions or values (e.g., temperature above/below zero, elevation above/below sea level, credits/debits, positive/negative electric charge); use positive and negative numbers to represent quantities in real-world contexts, explaining the meaning of 0 in each situation.
- 6.NS.C.6 Understand a rational number as a point on the number line. Extend number line diagrams and coordinate axes familiar from previous grades to represent points on the line and in the plane with negative number coordinates.

*2013-2014 Proposed Math Budget*

PK-5 Mathematics Budget

| Text/Materials<br>80,000 | Professional Development<br>$25,400 | Curriculum & Instruction Work<br>$20,260 |
|---|---|---|
| • Teacher resources<br>• Classroom materials | • PK-2 Trainer of trainers and outside trainers<br>• Building capacity on content knowledge<br>• Building capacity on pedagogy | • Update progress reports<br>• Develop PK-2 implementation guides<br>• Develop parent guide<br>• Revise homework<br>• K-2 revise and develop assessments to align with new standards |

6-12 Mathematics Budget

| Text/Materials<br>$120,000 | Professional Development<br>$15,000 | Curriculum & Instruction Work<br>$9,000 |
|---|---|---|
| • Textbooks and teacher resources | • Implementing the common core standards in the grades 6- 10 classrooms | • Curriculum writing for high school math courses following Geometry (grades 11-12)<br>• Develop supporting implementation guides and assessments for grades 6 – 10 aligned to the Common Core |