

## ALGEBRA TOPICS

### LAW OF EXPONENTS

- What you need to know:

Assume $a$ and $b$ are real numbers and $m$ and $n$ positive integers.		
(i) $a^m \cdot a^n = a^{m+n}$	(ii) $(ab)^m = a^m \cdot b^m$	(iii) $(a^m)^n = a^{mn}$
(iv) $a^{-m} = \frac{1}{a^m}$	(v) $a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$	

### PRACTICE PROBLEMS (NO CALCULATOR)

, please!)

Simplify:

1)  $(2c^2d^3)^3 = 2^3 c^6 d^9$

2)  $(a^2b)(-3ab^3)(-2ab)^2 = -\frac{3ab^2}{4}$

3)  $r^{n+2}(r^{k+1})^2 = r^{n+2} \cdot r^{2k+2} = r^{3n+4}$

4) Solve for  $n$ :  $3^{5n} = 3^5(3^{2n})^2$

5)  $4^{m+3} \cdot 16^n = 8^{m+2} \cdot 2^{4n}$   
 $2^{2m+6} \cdot 2^{4n} = 2^{m+6} \Rightarrow 2^{m+6} = 2^{m+6} \Rightarrow n=2$

$3^{5n} = 3^{5+4n}$

$\frac{3^{5n}}{3^5} = \frac{3^5}{3^5}$

$\frac{n}{5} = \frac{5}{5}$

$n = 5$

6) Write in exponential form (no negative exponents):  $\sqrt[3]{8x^6y^4} = \left(\frac{8x^6}{y^4}\right)^{\frac{1}{3}}$

7) Write in simplest radical form:  $\sqrt[3]{27} \cdot \sqrt[3]{9}$

$3^{\frac{3}{4}} \cdot 3^{\frac{3}{4}} = 3$

8) Simplify. Give your answer in exponential form:  $a^{\frac{1}{2}}(a^{\frac{3}{2}} - 2a^{\frac{1}{2}})$

$a^2 - 2a$

9) Using the property of exponents, show that  $27^{\frac{4}{3}} - 9^{\frac{5}{2}} = 2 \cdot 3^5$  (without the help of a calculator, of course)

$27^{\frac{4}{3}} - 9^{\frac{5}{2}} = (3^3)^{\frac{4}{3}} - (3^2)^{\frac{5}{2}} = 3^4 - 3^5 = 3^5(3-1) = 3^5 \cdot 2$

↓ Factor out  $3^5$ !

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Solve (algebraic solution):

10)  $(3x+1)^{\frac{3}{4}} = 8$

$3x+1 = 8^{\frac{4}{3}}$

$3x+1 = 2^4$

$x = 5$

11)  $9x^{-\frac{2}{3}} = 4$

$x^{-\frac{2}{3}} = \frac{4}{9}$

$x = \left(\frac{4}{9}\right)^{\frac{3}{2}}$

$x = \left(\frac{2^2}{3^2}\right)^{\frac{3}{2}}$

$x = \frac{8}{27}$

### FACTORING POLYNOMIALS

- What you need to know:

- Perfect Square Trinomials:	$a^2 + 2ab + b^2 = (a+b)^2$
	$a^2 - 2ab + b^2 = (a-b)^2$
- Difference of Squares:	$a^2 - b^2 = (a+b)(a-b)$
- Sum of Cubes:	$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
- Difference of Cubes:	$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

### PRACTICE PROBLEMS - Factor completely:

1)  $3x^2 - x - 10 = (3x+5)(x-2)$

2)  $4x^2 + 12x - 7 = (2x-1)(2x+7)$

3)  $a^2 - 10a + 25 = (a-5)^2$

4)  $4a^2 - 4ab + b^2 = (2a-b)^2$

5)  $25x^2 - 16y^2 = (5x-4y)(5x+4y)$

6)  $3x^2 - 48x + 3x(x^4 - 16) = 3x(x^2+4)(x^2-4) = 3x(x^2+4)(x+2)(x-2)$

7)  $x^6 - y^6 = (x^2 - y^2)(x^4 + x^2y^2 + y^4)$

8)  $64 - z^6 = (2^6 - z^2)(2^4 + 4z^2 + z^4) = (2^4 + z^2)(2^2 - z^2)(16 + 4z^2 + z^4)$

$= (x+y)(x-y)(x^2+y^2)(x^4+x^2y^2+y^4)$

9)  $8p^3 + 1 = (2p+1)(4p^2 - 2p + 1)$

10)  $x(y-2) + 3(2-y) = (y-2)(x-3)$

$= x(y-2) - 3(y-2) = (y-2)(x-3)$

11)  $x^2 - 6x + 9 - 4y^2$

$= (x-3)^2 - 4y^2 = (x-3+2y)(x-3-2y)$

13)  $6x^2 - 7xy - 3y^2 =$

$(3x+y)(2x-3y)$

12)  $(x+y)^2 + (x-y)^3$  Let  $x+y=a$ ,  $x-y=b$   
use sum of cubes formula above to get

$= 2x^3 [3y^2 + x^2]$

14)  $4x^3 + 8x^2y - 5xy^2$   
 $= x(4x^2 + 8xy - 5y^2)$   
 $= x(2x-y)(2x+5y)$

## SOLVING POLYNOMIAL INEQUALITIES

Example 1:

$$x^2 - 4x - 5 < 0$$

Solution:

$$(x+1)(x-5) < 0 \quad \text{factor}$$

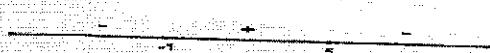


mark the zeros; pick a test point to determine the sign of the polynomial in each interval

The solution set of this conjunction is  $\{x : -1 < x < 5\}$ .

Example 2:  $-2x^2 + 8x + 10 < 0$

$$-2(x+1)(x-5) < 0 \quad \text{factor}$$



Make a sign chart; mark the zeros; pick a test point in each interval and plug it into the factored form to determine the sign of each interval

The solution set of this disjunction is:  $\{x : x \leq -1 \text{ or } x \geq 5\}$

## PRACTICE PROBLEMS

1)  $x^3 + 7x^2 + 10x > 0$

$$x(x^2 + 7x + 10) > 0$$

$$x(x+5)(x+2) > 0$$

sign chart: - + - +

Solution:  $\{x | -5 < x < -2 \text{ or } x > 0\}$

3)  $(x^2 - 4x)(x^2 - 4) \leq 0$

$$x(x-4)(x+2)(x-2) \leq 0$$

- + - + -

$\{x | -2 \leq x \leq 0 \text{ or } 2 \leq x \leq 4\}$

2)  $x^3 - 16x > 0$

$$x(x+4)(x-4) > 0$$

- + - +

Solution:  $\{x | -4 < x < 0 \text{ or } x > 4\}$

4)  $x^2 - x - 12 \geq 0$

$$(x-4)(x+3) \leq 0$$

+ - + -

$\{x | -3 \leq x \leq 4\}$

## PRE-CALCULUS TOPICS

### FUNCTIONS

You are given a polynomial equation and one or more of its roots. Find the remaining roots.

What you need to know:

- (i) **The Remainder Theorem:** When a polynomial  $P(x)$  is divided by  $(x-a)$ , the remainder is  $P(a)$ ;
  - (ii) **Factor Theorem:** For a polynomial  $P(x)$ ,  $(x-a)$  is a factor if and only if  $P(a)=0$ ;
  - (iii) Use synthetic division when dividing a polynomial by a linear factor; long division will always work;
  - (iv) **Odd Function:** A function  $f(x)$  is odd  $\Leftrightarrow f(-x) = -f(x)$ ;
  - (v) **Even Function:** A function  $f(x)$  is even  $\Leftrightarrow f(-x) = f(x)$ ;
- Remarks: item (iv) means the graph of the function has rotational symmetry about the origin, and (v) means the graph has reflectional symmetry about the y-axis;

### Puzzled? Let's try some...

Find the quotient and the remainder when the first polynomial is divided by the second. No calculators allowed!

1)  $x^3 - 2x^2 + 5x + 1 \div x - 1$

$$\begin{array}{r} 1 \quad -2 \quad 5 \quad 1 \\ \underline{-1} \quad -1 \quad 4 \quad 5 \\ \hline 1 \quad -1 \quad 4 \quad 5 \end{array}$$

Quotient:  $x^2 - x + 4$

Remainder: 5

3) Determine whether  $x-1$  or  $x+1$  are factors of  $x^{200} - 4x^{99} + 3$ .

$(x-1)$  is a Factor  $\Leftrightarrow P(1) = 0$

$$P(1) = 1 - 4 + 3 = 0 \implies (x-1) \text{ is a factor.}$$

$(x+1)$  is a Factor  $\Leftrightarrow P(-1) = 0$

$$P(-1) = 1 + 4 - 3 = 2 \neq 0 \implies (x+1) \text{ is not a factor.}$$

$$\begin{array}{r} 0 : 3x^2 - 3x + 2 \\ \rightarrow R : -41x + 1 \\ 2) 3x^4 - 2x^3 + 5x^2 + x + 1 \div x^2 + 2x \\ \hline x^2 + 2x \quad \begin{array}{r} 3x^2 - 8x + 2 \\ 3x^4 - 2x^3 + 5x^2 + x + 1 \\ - (3x^4 + 6x^3) \\ - 8x^3 + 5x^2 + x + 1 \\ - (-8x^3 - 16x^2) \\ \hline 21x^2 + x + 1 \\ - (21x^2 + 42x) \\ \hline - 41x + 1 \end{array} \end{array}$$

- 4) When a polynomial  $P(x)$  is divided by  $3x - 4$ , the quotient is  $x^3 + 2x^2 + 2$  and the remainder is  $-1$ . Find  $P(x)$ . (recall:  $P(x) = (\text{divisor}) \times (\text{quotient}) + \text{remainder}$ )

$$P(x) = (x^3 + 2x^2 + 2)(3x - 4) - 1$$

$$P(x) = 3x^4 - 4x^3 + 6x^2 - 8x + 6x - 8 - 1 \Rightarrow P(x) = 3x^4 - 4x^3 + 6x^2 - 2x - 9$$

- 5) You are given a polynomial equation and one or more of its roots. Find the remaining roots.  $2x^4 - 9x^3 + 2x^2 + 9x - 4 = 0$ ; roots:  $x = -1, x = 1$ .

Since  $(x-1)$  and  $(x+1)$  are factors  $\Rightarrow (x+1)(x-1) = x^2 - 1$  is a factor

Divide to find remaining roots:

$$\frac{2x^4 - 9x^3 + 2x^2 + 9x - 4}{x^2 - 1} \Rightarrow 2x^2 - 9x + 4 = (2x-1)(x-4)$$

other factors

$$\text{Remaining roots: } x = 1/2 \text{ and } x = 4$$

Verify algebraically whether the functions are odd or even.

$$6) f(x) = 5x^3 - x \quad 7) h(x) = \frac{5}{x^2 + 1}$$

$$f(-x) = 5(-x)^3 - (-x) \quad h(-x) = \frac{5}{(-x)^2 + 1} = \frac{5}{x^2 + 1} \quad \therefore \text{even!}$$

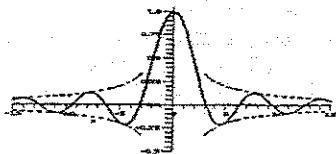
$$= -5x^3 + x$$

$$= -f(x) \quad \therefore \text{odd!}$$

### FINDING THE ASYMPTOTES OF A RATIONAL FUNCTION

What you need to know:

- Vertical Asymptote:** Set denominator = 0 and solve it. For values of  $x$  near the asymptote, the  $y$ -values of the function "approach" infinity (i.e., the  $y$ -values get increasingly large...)
- Horizontal or Oblique Asymptote:** when the values of  $x$  get very large (approach infinity), the  $y$ -values tend to the asymptote. Below is an example of a curve displaying asymptotic behavior:



A curve can intersect its asymptote, even infinitely many times. This does not apply to vertical asymptotes of functions.

- To find the horizontal or oblique asymptote, divide the numerator by the denominator. The asymptote is given by the quotient function.

### PRACTICE PROBLEMS (No please!)

- 1) For the functions below determine: a) the domain; b) the  $x$ -and  $y$ -intercepts; c) the vertical, horizontal or oblique asymptotes; and d) whether  $R(x)$  is even or odd (show algebraic analysis).

1)  $R(x) = \frac{3x^2 - 3x}{x^2 + x - 12}$     a)  $\{x \in \mathbb{R} \mid x \neq 3, x \neq -4\}$

b)  $x\text{-intercept: } x = 0, 1$

$y\text{-intercept: } (0, 0)$

c) V.A.:  $x = 3, x = -4$

H.A.:  $y = 3$

d) Neither even nor odd.

2)  $r(x) = \frac{3x^2 + 5x - 2}{x^2 - 4}$ . (explain what happens at  $x = -2$ )

a)  $\{x \in \mathbb{R} \mid x \neq \pm 2\}$

c) V.A.:  $x = 2$ , there is a "hole" at  $x = -2$

b)  $x\text{-intercept: } (\frac{1}{3}, 0)$

H.A.:  $y = 3$

$y\text{-intercept: } (0, \frac{1}{3})$

d) neither

- 3) Consider  $f(x) = \frac{8}{2+x^2}$ . a) is  $f$  even or odd? b) explain why there are no vertical asymptotes; c) What is the domain of  $f$ ? Range? d) Given the equation of its horizontal asymptote.

a)  $f(-x) = \frac{8}{2+(-x)^2} = \frac{8}{2+x^2} = f(x) \Rightarrow \text{even!}$

- b) Since the denominator  $2+x^2 \neq 0$  for every real number  
There are no V.A.'s.

c) Domain:  $\{x \in \mathbb{R}\}$   
Range:  $\{y \in \mathbb{R} \mid 0 < y \leq 4\}$  (the max  $y$ -value is attained when  $x = 0$ )

d)  $y = 0$



Solve the fractional equations. If it has no solution, say so...

$$8) \frac{3}{x+1} - \frac{1}{x-2} = \frac{1}{x^2-x-2}$$

$$\frac{3}{x+1} - \frac{1}{x-2} = \frac{1}{(x-2)(x+1)}$$

$$3(x-2) - 1(x+1) = 1$$

$$2x - 7 = 1$$

$$2x = 8 \Rightarrow x = 4$$

$$9) \frac{5}{a^2+a-6} = 2 - \frac{a-3}{a-2}$$

$$\frac{5}{(a+3)(a-2)} = \frac{2(a-2) - (a-3)}{(a-2)}$$

$$\frac{5}{(a+3)(a-2)} = \frac{a-1}{a-2}$$

$$5(a-2) = (a+3)(a-2)$$

$$5a-10 = (a+3)(a^2+a-6)$$

$$5a-10 = a^3 + a^2 - 6a - a^2 - a - 6$$

$$5a-10 = a^3 - 7a + 6$$

$$a^3 - 12a + 16 = 0$$

$$(a-2)(a^2 + 2a - 8) = 0 \quad (a-2)(a+4)(a-2)$$

$$a=2, a=-4$$

### EQUATIONS CONTAINING RADICALS

Example 1: Solve

$$3x - 5\sqrt{x} = 2$$

Isolate the radical term.

$$3x - 2 = 5\sqrt{x}$$

Square both sides

$$9x^2 - 12x + 4 = 25x$$

Solve for x

$$(x-4)(9x-1) = 0$$

$$x = 4 \text{ or } x = \frac{1}{9}$$

these are candidate solutions!

not possible

Now check each candidate above by plugging them into the original equation.  
The only possible solution is  $x = 4$ .

PRACTICE PROBLEMS: Solve. If an equation has no solution, say so. (Hint: sometimes you may have to square it twice...)

$$1) \sqrt{2x+5} - 1 = x$$

$$(\sqrt{2x+5})^2 = (x+1)^2$$

$$2x+5 = x^2 + 2x + 1$$

$$x^2 = 4$$

$$x = \pm 2$$

$\boxed{x=2}$  b/c  $x=-2$  is not a solution!

$$2) \sqrt{x-1} + \sqrt{x+4} = 5$$

$$\sqrt{x-1} = 5 - \sqrt{x+4}$$

$$x-1 = 25 - 10\sqrt{x+4} + x+4$$

$$-30 = -10\sqrt{x+4}$$

$$3 = \sqrt{x+4}$$

$$9 = x+4$$

$$\Rightarrow \boxed{x = \frac{5}{10}}$$

(Squaring both sides)

### FUNCTION OPERATIONS AND COMPOSITION: INVERSE FUNCTIONS

What you need to know:

- The composition of a function  $g$  with a function  $f$  is defined as:

$$h(x) = g(f(x))$$

The domain of  $h$  is the set of all  $x$ -values such that  $x$  is in the domain of  $f$  and  $f(x)$  is in the domain of  $g$ .

- Functions  $f$  and  $g$  are inverses of each other provided:

$$f(g(x)) = x \text{ and } g(f(x)) = x$$

The function  $g$  is denoted as  $f^{-1}$ , read as "f inverse".

- When a function  $f$  has an inverse, for every point  $(a,b)$  on the graph of  $f$ , there is a point  $(b,a)$  on the graph of  $f^{-1}$ . This means that the graph of  $f^{-1}$  can be obtained from the graph of  $f$  by changing every point  $(x,y)$  on  $f$  to the point  $(y,x)$ . This also means that the graphs of  $f$  and  $f^{-1}$  are reflections about the line  $y=x$ . Verify that the functions  $f(x) = x^3$  and  $g(x) = \sqrt[3]{x}$  are inverses!

- Horizontal Line Test: If any horizontal line which is drawn through the graph of a function  $f$  intersects the graph no more than once, then  $f$  is said to be a one-to-one function and has an inverse.



Let's try some problems!

Let  $f$  and  $g$  be functions whose values are given by the table below. Assume  $g$  is one-to-one with inverse  $g^{-1}$ .

$x$	$f(x)$	$g(x)$
1	6	2
2	9	3
3	10	4
4	-1	6

$$1) f(g(3)) = \boxed{-1}$$

$$2) g^{-1}(4) = \boxed{3}$$

$$3) f(g^{-1}(6)) = \boxed{-1}$$

$$4) g^{-1}(f(g(2))) = \boxed{3}$$

$$5) g(g^{-1}(2)) = \boxed{2}$$

\* y-intercept = 23  
means that when  $x=0$   
 $y=23$ . So...  
 $2(30)+a=23$

- 6) If  $f(x) = 2x^2 + 5$  and  $g(x) = 3x + a$ , find  $a$  so that the graph of  $f \circ g$  crosses the y-axis at 23.

$$f(g(x)) = f(3x+a) = 2(3x+a)^2 + 5$$

$$\begin{aligned} 2a^2 + 15 &= 23 \\ 2a^2 &= 18 \quad \boxed{a = \pm 3} \end{aligned}$$

## TRIGONOMETRY

What you need to know:

- Trig values for selected angles:

Radians	Degrees	$\sin x$	$\cos x$	$\tan x$
0	0°	0	1	0
$\pi/6$	30°	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
$\pi/4$	45°	$\sqrt{2}/2$	$\sqrt{2}/2$	1
$\pi/3$	60°	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$\pi/2$	90°	1	0	Und.

- Key Trig Identities:

$$\begin{aligned} (\text{i}) \quad \sin^2 \theta + \cos^2 \theta &= 1 \\ (\text{ii}) \quad \tan^2 \theta + 1 &= \sec^2 \theta \\ (\text{iv}) \quad 1 + \cot^2 \theta &= \csc^2 \theta \\ (\text{v}) \quad \sin(x+y) &= \sin x \cos y + \cos x \sin y \\ (\text{vi}) \quad \sin(x-y) &= \sin x \cos y - \cos x \sin y \\ (\text{vii}) \quad \cos(x+y) &= \cos x \cos y - \sin x \sin y \\ (\text{viii}) \quad \cos(x-y) &= \cos x \cos y + \sin x \sin y \\ (\text{ix}) \quad \sin 2\theta &= 2 \sin \theta \cos \theta \\ (\text{x}) \quad \cos 2\theta &= \cos^2 \theta - \sin^2 \theta \text{ or} \\ &= 2 \cos^2 \theta - 1 \\ &= 1 - 2 \sin^2 \theta \end{aligned}$$

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Without using a calculator or table, find each value:

1.  $\cos(-\pi)$

2.  $\sin \frac{3\pi}{2}$

3.  $\cos \frac{7\pi}{6}$

4.  $\sin \frac{11\pi}{6}$

5.  $\cos \frac{5\pi}{4}$

6.  $\sin \left(-\frac{2\pi}{3}\right)$

7. Find the slope and equation of the line with y-intercept = 4 and inclination  $45^\circ$ . (the inclination of a line is the angle  $\alpha$ , where  $0^\circ \leq \alpha \leq 180^\circ$ , that is measured from the positive x-axis to the line. If a line has slope  $m$ , then  $m = \tan \alpha$ .)

$m = 1$  through  $(0, 4)$

$y = x + 4$

8. Find the inclination of the line given the equation:  $3x + 5y = 8$ . (you'll need a calculator here...)

$$\begin{aligned} 5y &= -3x + 8 \\ y &= -\frac{3}{5}x + \frac{8}{5} \end{aligned}$$

$$\tan \alpha = -\frac{3}{5} \Rightarrow \alpha = \tan^{-1} \left( \frac{3}{5} \right)$$

$\alpha = 14.9^\circ$

Solve the trigonometric equations algebraically by using identities and without the use of a calculator. You must find all solutions in the interval  $0 \leq \theta \leq 2\pi$ :

9.  $2\sin^2 \theta - 1 = 0$

$$\sin^2 \theta = \frac{1}{2}$$

$$\sin \theta = \pm \frac{\sqrt{2}}{2}$$

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}$$

11.  $\sin^2 x = \sin x$

$$\sin^2 x - \sin x = 0$$

$$\sin x(\sin x - 1) = 0$$

$$\sin x = 0 \text{ or } \sin x = 1$$

13.  $2\cos^2 x + \cos x - 1 = 0$

$$(2\cos x + 1)(\cos x - 1) = 0$$

$$\cos x = -\frac{1}{2} \text{ or } \cos x = 1$$

$$x = \frac{2\pi}{3}, \pi, \frac{4\pi}{3}$$

15.  $\sin 2x = \cos x$

$$\sin^2 x - \cos^2 x = 0$$

$$\sin x(\sin x - \cos x) = 0$$

$$\sin x = 0 \text{ or } \sin x = \cos x$$

$$x = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$

10.  $\cos x \tan x = \cos x$

$$\cos x \frac{\sin x}{\cos x} = \cos x$$

$$\sin x = \cos x \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$$

12.  $2\cos^2 x = \cos x$

$$\cos x(2\cos x - 1) = 0$$

$$\cos x = 0 \text{ or } \cos x = \frac{1}{2}$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

14.  $\sin 2x = \cos x$

$$\sin^2 x - \cos^2 x = 0$$

$$\sin x(\sin x - \cos x) = 0$$

$$\sin x = 0 \text{ or } \sin x = \cos x$$

$$x = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

## EXPONENTIAL AND LOGARITHMIC FUNCTIONS

### LOGARITHMS - What you need to know...

- Definition of Logarithm with base b: For any positive numbers b and y with  $b \neq 1$ , we define the logarithm of y with base b as follows:  
 $\log_b y = x$  if and only if  $b^x = y$
- Laws of Logarithms (for  $M, N, b > 0, b \neq 1$ )
  - (I)  $\log_b MN = \log_b M + \log_b N$
  - (II)  $\log_b \frac{M}{N} = \log_b M - \log_b N$
  - (III)  $\log_b M = \log_b N$  if and only if  $M = N$
  - (IV)  $\log_b M^k = k \cdot \log_b M$
  - (V)  $b^{\log_b M} = M$
- The logarithmic and exponential functions are inverse functions.  
 Example: Consider  $f(x) = 2^x$  and  $g(x) = \log_2 x$ . Verify that (3, 8) is on the graph of f and (8, 3) is on the graph of g. In addition,  
 $f(g(x)) = 2^{\log_2 x} = x$  and  $g(f(x)) = \log_2(2^x) = x$  which verifies that f and g are indeed inverse functions.

Evaluate. The first is to be used as an example. (No calculators, please!)

1.  $\log_4 64 = 3$  (because  $4^3 = 64$ )

2.  $\log_5 \frac{1}{125} = -3$

3.  $\log_4 \sqrt[4]{4} = \frac{1}{4}$

4.  $\log_{\frac{1}{3}} 9 = -2$

5.  $\log_{\sqrt{5}} 125 = 6$

6.  $\log_{31} 243 = \frac{5}{4}$  (solve  $31^x = 243 \Rightarrow 3^{4x} = 3^5 \Rightarrow 4x = 5 \Rightarrow x = \frac{5}{4}$ )

Solve for x.

7.  $\log_x 64 = 3$

$x = 4$

8.  $\log_3(x^2 - 7) = 2$

$|x = \pm 4|$

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9.  $\log_3(\log_2 x) = 0$

$x = 3$

10.  $\log_6(\log_4(\log_2 x)) = 0$

$\log_4(\log_2 x) = 6^0 = 1 \Rightarrow \log_2 x = 4 \Rightarrow x = 16$

11.  $\log_2(\log_x 64) = 1$

$\log_x 64 = 2 \Rightarrow |x = 8|$

12. Show that  $\log_2 4 + \log_2 8 = \log_2 32$  by evaluating the 3 logarithms.

Make a generalization based on your findings.

$\log_2 4 = 2$       Thus, The sum of 2 logs = log of a product  
 $\log_2 8 = 3$       or       $\log_2 A + \log_2 B = \log_2 A \cdot B$   
 $\log_2 32 = 5$

Write the given expression as a rational number or as a single logarithm.  
 (use the laws of logarithms above!)

13.  $\log 8 + \log 5 - \log 4$

$\log 10 = 1$

14.  $\log_2 48 - \frac{1}{3} \log_2 27 = \log_2 16 = 4$

15.  $\frac{1}{3}(2\log M - \log N - \log P)$

$\frac{1}{3}(\log \frac{M^2}{NP})$

16.  $\log_8 \sqrt{80} - \log_8 \sqrt{5} = \log_8 \frac{\sqrt{80}}{\sqrt{5}} = \log_8 \frac{\sqrt{160}}{\sqrt{5}} = \log_8 4$

If  $\log_8 3 = r$  and  $\log_8 5 = s$ , express the given logarithm in terms of r and s.

17.  $\log_8 75 = r + s$

$\therefore \log_8 75 = \log_8(3 \cdot 25)$

18.  $\log_8 0.12 = r - 2s$

$\therefore \log_8 \frac{12}{100} = \log_8 \frac{3}{25} = \log_8 3 - \log_8 5^2$

19. Solve for x:  $\log_2(x^2 + 8) = \log_2 x + \log_2 6$

$\log_2(x^2 + 8) = \log_2 6x \Rightarrow x^2 + 8 = 6x \Rightarrow x^2 - 6x + 8 = 0 \Rightarrow (x-4)(x-2) = 0 \Rightarrow |x=4| \quad |x=2|$

Solve (no calculator please!)

20.  $3^x = 81$

$x = 4$

Use the fact  
 $3^x = 3^4$

21.  $4^x = 8$

$x = \frac{3}{2}$

change into:  
 $2^{2x} = 2^3$

22.  $4^x = 16\sqrt{2}$

$x = \frac{9}{4}$

change into:  
 $2^{2x} = 2^{\frac{9}{2}}$

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23.  $2^{2x} - 2^x - 6 = 0$  (hint:  $2^{2x} - 2^x - 6 = (2^x)^2 - (2^x) - 6$ ) Treat this as a quadratic equation:  
 $X = \frac{\ln 3}{\log 2}$  change into:  $u^2 - u - 6 = 0$   
 $(u-3)(u+2) = 0 \Rightarrow u=3 \text{ or } u=-2$  not possible. If  $u=3 \Rightarrow 2^x=3 \Rightarrow \log 2^x = \log 3 \Rightarrow x = \frac{\ln 3}{\log 2}$

Simplify (recall: the notation. ln stands for the natural logarithm base e).  
24. a)  $\ln e^x = x$  b)  $e^{\ln x} = x$  c)  $e^{2\ln x} = x^2$  d)  $e^{-\ln x} = x^{-1} \text{ or } \frac{1}{x}$

Solve for x (answers can be left in terms of e).

25.  $\ln x - \ln(x-1) = \ln 2$

$X = 2$   
Solve:  $\ln \frac{x}{x-1} = \ln 2$

26.  $(\ln x)^2 - 6(\ln x) + 9 = 0$

Let  $u = \ln x$   
 $u^2 - 6u + 9 = 0 \Rightarrow (u-3)^2 = 0$   
 $u=3 \Rightarrow \ln x = 3 \Rightarrow x = e^3$

We conclude with the following definition:

#### EXPONENTIAL GROWTH MODEL:

Consider an initial quantity  $P_0$  that changes exponentially with time t. At any time t the amount Y after t units of time may be given by:

$$Y = P_0 e^{kt}$$

Where  $k > 0$  represents the growth rate and  $k < 0$  represents the decay rate.

#### 27. Growth of Cholera

Suppose that the cholera bacteria in a colony grows according to the exponential model  $P = P_0 e^{kt}$ . The colony starts with 1 bacterium and doubles in number every half hour. How many bacteria will the colony contain at the end of 24 hours?

Step 1: Find the growth rate k

$$2 = 1 \cdot e^{.5k}$$

$$\ln 2 = .5k \Rightarrow k = \frac{\ln 2}{.5}$$

Step 2:  $\frac{\ln 2}{.5} (24)$

$$P = e^{\frac{\ln 2}{.5} (24)}$$

$$P = 2.815 \times 10^{14} \leftarrow \text{number of bacteria}$$

#### 28. Bacteria Growth

A colony of bacteria is grown under ideal conditions in a laboratory so that the population increases exponentially with time. At the end of 3 hours

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there are 10,000 bacteria. At the end of 5 h there are 40,000 bacteria. How many bacteria were present initially?

From condition 1:  $10,000 = P_0 e^{5k} \Rightarrow P_0 = \frac{10,000}{e^{5k}} \quad \text{①}$

From condition 2:  $40,000 = P_0 e^{5k} \Rightarrow P_0 = \frac{40,000}{e^{5k}} \quad \text{②}$

\* Substituting in either ① or ②

$$P_0 = \frac{10,000}{e^{5k}} = 1250 \text{ bacteria}$$

Thus:  $\frac{10,000}{e^{5k}} = \frac{40,000}{e^{5k}} \Rightarrow \frac{e^{5k}}{e^{5k}} = \frac{40,000}{10,000} \Rightarrow e^{5k} = 4 \Rightarrow 5k = \ln 4 \Rightarrow k = \frac{\ln 4}{5}$

The half-life of a radioactive element is the time required for half of the radioactive nuclei present in a sample to decay. The next problem shows that the half-life is a constant that depends only on the radioactive substance and not on the number of radioactive nuclei present in the sample.

29) Find the half-life of a radioactive substance with decay equation  $y = y_0 e^{-kt}$  and show that the half-life depends only on k. (hint: set up

the equation  $y_0 e^{-kt} = \frac{1}{2} y_0$ . Solve it algebraically for t!)

Solve  $\frac{1}{2} = e^{-kt} \Rightarrow \ln \frac{1}{2} = -kt \Rightarrow t = \left[ \ln \frac{1}{2} \right] \left( -\frac{1}{k} \right) \Rightarrow$

$$t = \frac{\ln 2}{k}$$

Conclusion: The half-life of a radioactive substance with rate constant k ( $k > 0$ ) is

$$\text{Half-life} = \frac{\ln 2}{k}$$

#### 30) Using Carbon-14 Dating

Scientists who do carbon-14 dating use 5700 years for its half-life. Find the age of a sample in which 10% of the radioactive nuclei originally present have decayed.

Since  $5700 = \text{half-life} \Rightarrow 5700 = \frac{\ln 2}{k} \Rightarrow k = \frac{\ln 2}{5700}$

10% decayed  $\Rightarrow 90\% \text{ of radioactive nuclei still present}$

Thus, solve  $.9 = e^{\frac{\ln 2}{5700} t}$

$$\ln .9 = \frac{\ln 2}{5700} t$$

$$t = \frac{5700}{\ln 2} \cdot \ln (.9)$$

{ Age is 366.4 years