

SUMMER PREP FOR AP CALCULUS AB/BC



With topics from Algebra and Pre-Calculus to

get you ready to  the AP!
(Key contains solved problems)

Note: The purpose of this packet is to give you a review of basic skills. You are asked not to use the calculator, except on p.13 (8) and p.16-17(27-30all).

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ALGEBRA TOPICS

LAWS OF EXPONENTS

- What you need to know:

Assume **a** and **b** are real numbers and **m** and **n** positive integers.

(i) $a^m \cdot a^n = a^{m+n}$

(ii) $(ab)^m = a^m \cdot b^m$

(iii) $(a^m)^n = a^{mn}$

(iv) $a^{-m} = \frac{1}{a^m}$

(v) $a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$

PRACTICE PROBLEMS (NO , please!)

Simplify:

1) $(2c^2d^3)^3$

2) $(a^2b)(-3ab^3)(-2ab)^{-2}$

3) $r^{h-2}(r^{h+1})^2$


4) Solve for n: $3^{5n} = 3^5(3^{2n})^2$

5) $4^{n+3} \cdot 16^n = 8^{3n}$

6) Write in exponential form (no negative exponents): $\sqrt[3]{8x^6y^{-4}}$

7) Write in simplest radical form: $\sqrt[4]{27} \cdot \sqrt[8]{9}$

8) Simplify. Give your answer in exponential form: $a^{\frac{1}{2}} \left(a^{\frac{3}{2}} - 2a^{\frac{1}{2}} \right)$

9) Using the property of exponents, show that $27^{\frac{4}{3}} - 9^{\frac{3}{2}} = 2 \cdot 3^3$ (without the help of a  , of course!)

Solve (algebraic solution!):

10) $(3x + 1)^{\frac{3}{4}} = 8$

11) $9x^{-\frac{2}{3}} = 4$

FACTORING POLYNOMIALS

What you need to know:

- **Perfect Square Trinomials:** $a^2 + 2ab + b^2 = (a + b)^2$
 $a^2 - 2ab + b^2 = (a - b)^2$
- **Difference of Squares:** $a^2 - b^2 = (a + b)(a - b)$
- **Sum of Cubes:** $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
- **Difference of Cubes:** $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

PRACTICE PROBLEMS - Factor completely:

1) $3x^2 - x - 10$

2) $4x^2 + 12x - 7$

3) $a^2 - 10a + 25$

4) $4a^2 - 4ab + b^2$

5) $25x^2 - 16y^2$

6) $3x^5 - 48x$

7) $x^6 - y^6$

8) $64 - z^6$

9) $8p^3 + 1$

10) $x(y - 2) + 3(2 - y)$

11) $x^2 - 6x + 9 - 4y^2$

12) $(x + y)^3 + (x - y)^3$

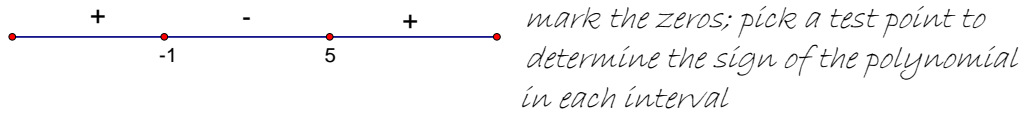
13) $6x^2 - 7xy - 3y^2$

14) $4x^3 + 8x^2y - 5xy^2$

SOLVING POLYNOMIAL INEQUALITIES

Example 1: $x^2 - 4x - 5 < 0$

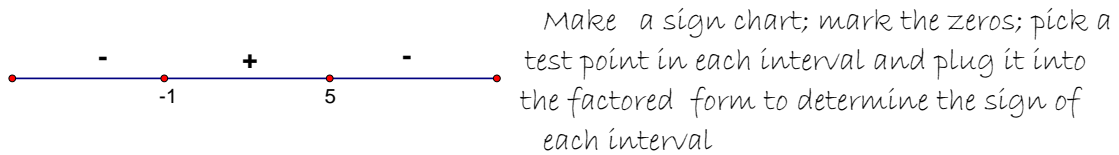
Solution: $(x + 1)(x - 5) < 0$ factor



The solution set of this **conjunction** is $\{x : -1 < x < 5\}$.

Example 2: $-2x^2 + 8x + 10 < 0$

$-2(x + 1)(x - 5) < 0$ Factor



The solution set of this **disjunction** is: $\{x : x < -1 \text{ or } x > 5\}$

PRACTICE PROBLEMS

1) $x^3 + 7x^2 + 10x > 0$

2) $x^3 - 16x > 0$

3) $(x^2 - 4x)(x^2 - 4) \leq 0$

4) $x^2 - x - 12 < 0$

PRE-CALCULUS TOPICS

FUNCTIONS

You are given a polynomial equation and one or more of its roots. Find the remaining roots.

What you need to know:

- (i) **The Remainder Theorem:** When a polynomial $P(x)$ is divided by $(x-a)$, the remainder is $P(a)$;
 - (ii) **Factor Theorem:** For a polynomial $P(x)$, $(x-a)$ is a factor if and only if $P(a)=0$;
 - (iii) Use **synthetic** division when dividing a polynomial by a linear factor; long division will always work;
 - (iv) **Odd Function:** A function $f(x)$ is odd $\Leftrightarrow f(-x) = -f(x)$;
 - (v) **Even Function:** A function $f(x)$ is even $\Leftrightarrow f(-x) = f(x)$;
- Remarks: item (iv) means the graph of the function has **rotational symmetry** about the origin, and (v) means the graph has **reflectional symmetry** about the y-axis;

Puzzled  **Let's try some...**

Find the quotient and the remainder when the first polynomial is divided by the second. No calculators allowed!

1) $x^3 - 2x^2 + 5x + 1$; $x - 1$

2) $3x^4 - 2x^3 + 5x^2 + x + 1$; $x^2 + 2x$

3) Determine whether $x-1$ or $x+1$ are factors of $x^{100} - 4x^{99} + 3$.

4) When a polynomial $P(x)$ is divided by $3x - 4$, the quotient is $x^3 + 2x + 2$ and the remainder is -1 . Find $P(x)$. (recall: $P(x) = (\text{divisor}) \times (\text{quotient}) + \text{remainder}$!)

5) You are given a polynomial equation and one or more of its roots. Find the remaining roots. $2x^4 - 9x^3 + 2x^2 + 9x - 4 = 0$; roots: $x = -1, x = 1$.

Verify algebraically whether the functions are odd or even.

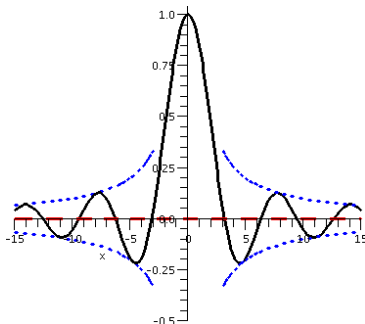
6) $f(x) = 5x^3 - x$

7) $h(x) = \frac{5}{x^2 + 1}$

FINDING THE ASYMPTOTES OF A RATIONAL FUNCTION

What you need to know:

- **Vertical Asymptote:** Set denominator = 0 and solve it. For values of x near the asymptote, the y -values of the function "approach" infinity (i.e., the y -values get increasingly large...)
- **Horizontal or Oblique Asymptote:** when the values of x get very large (approach infinity), the y -values tend to the asymptote. Below is an example of a curve displaying asymptotic behavior:



A curve can intersect its asymptote, even infinitely many times. This **does not apply** to vertical asymptotes of functions.

- To find the horizontal or oblique asymptote, divide the numerator by the denominator. The asymptote is given by the quotient function.

PRACTICE PROBLEMS (No , please!)

- 1) For the functions below determine: a) the domain; b) the x-and y-intercepts; c) the vertical, horizontal or oblique asymptotes; and d) whether $R(x)$ is even or odd (show algebraic analysis).

1) $R(x) = \frac{3x^2 - 3x}{x^2 + x - 12}$

2) $r(x) = \frac{3x^2 + 5x - 2}{x^2 - 4}$. (explain what happens at $x = -2$...)

- 3) Consider $f(x) = \frac{8}{2 + x^2}$. a) is f even or odd? b) explain why there are no vertical asymptotes; c) What is the domain of f ? Range? d) Given the equation of its horizontal asymptote.

4) Find a quadratic function $f(x)$ with y-intercept -2 and x-intercepts $(-2,0)$ and $(\frac{1}{3},0)$.

RATIONAL ALGEBRAIC EXPRESSIONS AND FUNCTIONS

What you need to know:

(i) A rational function is a function of the form $\frac{p(x)}{q(x)}$, where p and q are polynomial functions and $q(x) \neq 0$ for every x .

(ii) To **simplify** a rational expression, factor both numerator and denominator. Cancel out common factors.

(iii) To find the **domain** of a rational function: solve $q(x) = 0$. The zeroes of the polynomial $q(x)$ must be excluded from the domain.

(iv) To find the **zeroes** of a rational function: set $p(x) = 0$ and solve it. The zeros of $p(x)$ are the zeros of the polynomial function (provided they are not also zeros of the denominator $q(x)$!)

In #1-3: Find (a) the domain of each function; and (b) the zeros, if any. No calculators, please!

$$1) g(x) = \frac{2x^2 + 3x - 9}{x^3 - 4x}$$

$$2) g(x) = \frac{(x+3)(x-2)(x-1)}{x^2 - 6x + 8}$$

$$3) h(x) = \frac{x^3 + x^2 - x - 1}{x^3 - x^2 - x + 1}$$

Simplify: (attention: just perform the indicated operation...)

$$4) \frac{3}{x^2 - 5x + 6} + \frac{2}{x^2 - 4}$$

$$5) \frac{x^2}{x-1} \cdot \frac{x+1}{x+2} \div \frac{x}{(x-1)(x+2)}$$

$$6) \frac{\frac{x^2 - y^2}{x + y}}{x^4 - y^4} =$$

Find the constants A and B that make the equation true.

$$7) \frac{2x - 9}{x^2 - x - 6} = \frac{A}{x - 3} + \frac{B}{x + 2}$$

Solve the fractional equations. If it has no solution, say so...

$$8) \frac{3}{x+1} - \frac{1}{x-2} = \frac{1}{x^2 - x - 2}$$

$$9) \frac{5}{a^2 + a - 6} = 2 - \frac{a-3}{a-2}$$

EQUATIONS CONTAINING RADICALS

Example 1: Solve

$$3x - 5\sqrt{x} = 2$$

$$3x - 2 = 5\sqrt{x} \quad \text{isolate the radical term}$$

$$9x^2 - 12x + 4 = 25x \quad \text{square both sides}$$

$$9x^2 - 37x + 4 = 0 \quad \text{Solve for } x$$

$$(x-4)(9x-1) = 0$$

$$x = 4 \text{ or } x = \frac{1}{9} \quad \text{these are candidate solutions!}$$

Now check each candidate above by plugging them into the original equation.
The only possible solution is $x = 4$.

PRACTICE PROBLEMS: Solve. If an equation has no solution, say so. (hint: sometimes you may have to square it twice...)

$$1) \sqrt{2x+5} - 1 = x$$

$$2) \sqrt{x-1} + \sqrt{x+4} = 5$$

FUNCTION OPERATIONS AND COMPOSITION; INVERSE FUNCTIONS

What you need to know:

- The **composition** of a function g with a function f is defined as:

$$h(x) = g(f(x))$$

The domain of h is the set of all x -values such that x is in the domain of f and $f(x)$ is in the domain of g .

- Functions f and g are inverses of each other provided:

$$f(g(x)) = x \text{ and } g(f(x)) = x$$

The function g is denoted as f^{-1} , read as "f inverse".

- When a function f has an inverse, for every point (a,b) on the graph of f , there is a point (b,a) on the graph of f^{-1} . This means that the graph of f^{-1} can be obtained from the graph of f by changing every point (x,y) on f to the point (y,x) . This also means that the graphs of f and f^{-1} are reflections about the line $y=x$. Verify that the functions $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ are inverses!
- Horizontal Line Test: If any horizontal line which is drawn through the graph of a function f intersects the graph no more than once, then f is said to be a **one-to-one** function and has an inverse.



Let's try some problems!

Let f and g be functions whose values are given by the table below. Assume g is one-to-one with inverse g^{-1} .

x	$f(x)$	$g(x)$
1	6	2
2	9	3
3	10	4
4	-1	6

1) $f(g(3))$

2) $g^{-1}(4)$

3) $f(g^{-1}(6))$

4) $f^{-1}(f(g(2)))$

5) $g(g^{-1}(2))$

6) If $f(x) = 2x^2 + 5$ and $g(x) = 3x + a$, find a so that the graph of $f \circ g$ crosses the y -axis at 23.

TRIGONOMETRY

What you need to know:

- Trig values for selected angles:

Radians	Degrees	Sin x	Cos x	Tan x
0	0°	0	1	0
$\frac{\pi}{6}$	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	90°	1	0	Und.

- Key Trig Identities - formulas you should know:

(i) $\sin^2 \theta + \cos^2 \theta = 1$

(ii) $\tan^2 \theta + 1 = \sec^2 \theta$

(iv) $1 + \cot^2 \theta = \csc^2 \theta$

(v) $\sin(x + y) = \sin x \cos y + \cos x \sin y$

(vi) $\sin(x - y) = \sin x \cos y - \cos x \sin y$

(vii) $\cos(x + y) = \cos x \cos y - \sin x \sin y$

(viii) $\cos(x - y) = \cos x \cos y + \sin x \sin y$

(ix) $\sin 2\theta = 2 \sin \theta \cos \theta$

(x) $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$ or

$$= 2 \cos^2 \theta - 1$$

$$= 1 - 2 \sin^2 \theta$$

Without using a calculator or table, find each value:

1. $\cos(-\pi)$

2. $\sin\frac{3\pi}{2}$

3. $\cos\frac{7\pi}{6}$

4. $\sin\frac{11\pi}{6}$

5. $\cos\frac{5\pi}{4}$

6. $\sin\left(-\frac{2\pi}{3}\right)$

7. Find the slope and equation of the line with y-intercept = 4 and ***inclination*** 45° . (the ***inclination*** of a line is the angle α , where $0^\circ \leq \alpha \leq 180^\circ$, that is measured from the positive x-axis to the line. If a line has slope m , then $m = \tan \alpha$).

8. Find the inclination of the line given the equation: $3x + 5y = 8$ (*you'll need a calculator here...*)

Solve the trigonometric equations algebraically by using identities and **without** the use of a calculator. You must find all solutions in the interval $0 \leq \theta \leq 2\pi$:

9. $2\sin^2 \theta - 1 = 0$

10. $\cos x \tan x = \cos x$

11. $\sin^2 x = \sin x$

12. $2\cos^2 x = \cos x$

13. $2\cos^2 x + \cos x - 1 = 0$

14. $\sin 2x = \cos x$

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

LOGARITHMS - What you need to know...

- **Definition of Logarithm with base b:** For any positive numbers b and y with $b \neq 1$, we define the **logarithm of y with base b** as follows:

$$\log_b y = x \text{ if and only if } b^x = y$$

- **LAWS OF LOGARITHMS** (for $M, N, b > 0, b \neq 1$)

$$(i) \log_b MN = \log_b M + \log_b N$$

$$(ii) \log_b \frac{M}{N} = \log_b M - \log_b N$$

$$(iii) \log_b M = \log_b N \text{ if and only if } M = N$$

$$(iv) \log_b M^k = k \cdot \log_b M$$

$$(v) b^{\log_b M} = M$$

- The logarithmic and exponential functions are **inverse** functions.
Example: Consider $f(x) = 2^x$ and $g(x) = \log_2 x$. Verify that (3,8) is on the graph of f and (8,3) on the graph of g . In addition,

$f(g(x)) = 2^{\log_2 x} = x$ and $g(f(x)) = \log_2(2^x) = x$ which verifies that f and g are indeed inverse functions.

Evaluate. The first is to be used as an example. (No Calculators, please!)

1. $\log_4 64 = 3$ (because $4^3 = 64$)

2. $\log_5 \frac{1}{125} =$

3. $\log_4 \sqrt[4]{4} =$

4. $\log_{\frac{1}{3}} 9 =$

5. $\log_{\sqrt{5}} 125 =$

6. $\log_{81} 243 =$

Solve for x .

7. $\log_x 64 = 3$

8. $\log_3(x^2 - 7) = 2$

9. $\log_5(\log_3 x) = 0$

10. $\log_6(\log_4(\log_2 x)) = 0$

11. $\log_2(\log_x 64) = 1$

12. Show that $\log_2 4 + \log_2 8 = \log_2 32$ by evaluating the 3 logarithms. Make a generalization based on your findings.

Write the given expression as a rational number or as a single logarithm. (Use the laws of logarithms above!)

13. $\log 8 + \log 5 - \log 4$

14. $\log_2 48 - \frac{1}{3} \log_2 27$

15. $\frac{1}{3}(2\log M - \log N - \log P)$

16. $\log_8 \sqrt{80} - \log_8 \sqrt{5}$

If $\log_8 3 = r$ and $\log_8 5 = s$, express the given logarithm in terms of r and s .

17. $\log_8 75$

18. $\log_8 0.12$

19. Solve for x : $\log_2(x^2 + 8) = \log_2 x + \log_2 6$

Solve (no calculator please!)

20. $3^x = 81$

21. $4^x = 8$

22. $4^x = 16\sqrt{2}$

23. $2^{2x} - 2^x - 6 = 0$ (hint: $2^{2x} - 2^x - 6 = (2^x)^2 - (2^x) - 6$)

Simplify (recall: the notation \ln stands for the natural logarithm base e)

24. a) $\ln e^x$ b) $e^{\ln x}$ c) $e^{2\ln x}$ d) $e^{-\ln x}$

Solve for x (answers can be left in terms of e).

25. $\ln x - \ln(x - 1) = \ln 2$

26. $(\ln x)^2 - 6(\ln x) + 9 = 0$

We conclude with the following definition:

EXPONENTIAL GROWTH MODEL:

Consider an initial quantity Y_0 that changes exponentially with time t .
At any time t the amount Y after t units of time may be given by:

$$Y = Y_0 e^{kt}$$

Where $k > 0$ represents the **growth rate** and $k < 0$ represents the **decay rate**.

27. Growth of Cholera

Suppose that the cholera bacteria in a colony grows according to the exponential model $P = P_0 e^{kt}$. The colony starts with 1 bacterium and doubles in number every half hour. How many bacteria **will** the colony contain at the end of 24 hours?

28. Bacteria Growth

A colony of bacteria is grown under ideal conditions in a laboratory so that the population increases exponentially with time. At the end of 3 hours

there are 10,000 bacteria. At the end of 5 h there are 40,000 bacteria. How many bacteria were present initially?

The **half-life** of a radioactive element is the time required for half of the radioactive nuclei present in a sample to decay. The next problem shows that the half-life is a **constant** that depends only on the radioactive substance and not on the number of radioactive nuclei present in the sample.

29) Find the half-life of a radioactive substance with decay equation $y = y_0 e^{-kt}$ and show that the half-life depends only on k . (hint: set up the equation $y_0 e^{-kt} = \frac{1}{2} y_0$. Solve it algebraically for t !)

Conclusion: The half-life of a radioactive substance with rate constant k ($k > 0$) is

Half-life = _____

30) Using Carbon-14 Dating

Scientists who do carbon-14 dating use 5700 years for its half-life. Find the age of a sample in which 10% of the radioactive nuclei originally present have decayed.